Evelyn Fernandes Erickson

An Investigation of Logical Pluralism and B-entailment

Natal, RN, Brasil

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Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Filosofia como requisito parcial para obtenção do título de Mestre em Filosofia.

Universidade Federal do Rio Grande do Norte – UFRN

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Prof. Dr. João Marcos de Almeida
Orientador

Prof. Dr. Daniel Durante Pereira Alves
Membro Interno

Prof. Dr. Abílio Azambuja Rodrigues Filho
Membro Externo

Natal, RN, Brasil
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To my parents, for their continued incentive and support.

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"A chain of valid reasoning can end only with the determination of truth, and I'll stick till I get there."

— Isaac Asimov, *I Robot*

Resumo

O pluralismo lógico vem recentemente chamando atenção, com vários autores contestando sua natureza. O pluralismo lógico é a posição que diz que há mais de uma lógica correta ou legítima, o que pode ser articulado de diferentes maneiras. A presente dissertação participa nesse debate explorando o framework do B-entailment no contexto de variedades do pluralismo presentes na literatura. O B-entailment é uma noção de consequência lógica que é capaz de expressar outras relações, como as lógicas multi-dimensionais. Em particular, esse estudo examina quatro posições sobre o pluralismo: o pluralismo eclético de Shapiro, o pluralismo através de GTT de Beall e Restall, o pluralismo intra-teórico de Hjortland e os pluralismos através de teoria das demonstações de Restall e Paoli. Será mostrado como o B-entailment se encaixa nessas variedades de pluralismo e também como fica em falta em certos aspectos. O objetivo da dissertação é tanto contribuir para a discussão sobre o pluralismo lógico quanto expandir a discussão sobre o B-entailment e outras noções de consequência lógica desse gênero.

Palavras-chaves: Pluralismo lógico. Lógica multivalorada. B-entailment. Relação de consequência.

Abstract

Logical pluralism has recently been given much attention, with numerous authors contesting its nature. In a broad sense, logical pluralism is the view that there are more than one correct or legitimate logic, which can be articulated in different ways. The present study participates in this debate by exploring the framework of B-entailment in the context of several of varieties of pluralism. B-entailment is a logical consequence relation, which may express different consequence relations, such as those of many-dimensional logics. In particular, this study examines four different takes on pluralism: Shapiros's eclectic pluralism, Beall and Restall's GTT pluralism, Hjortland's intra-theoretic pluralism and Restall and Paoli's pluralism based on proof-theory. It will be shown how B-entailment fits into these varieties of pluralism, and a discussion is presented on how B-entailment falls short is some aspects. The aim is to contribute to the discussion on logical pluralism as well as to expand the discussion about B-entailment and consequence relations of this sort.

Key-words: Logical pluralism. Multivalued logic. B-entailment. Consequence relation.

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Preface

Logical pluralism is a topic gaining momentum in the literature on Philosophy of Logic. It is an attempt to conciliate developments on non-classical logics with classical logic, and to do so one must be open to accepting that more than one logic is correct. This acceptance can mean many things and there are numerous suggestions on how this might be done. The thesis starts out by clearing up these issues and relies on the taxonomies of pluralism presented by Graham Priest [Priest, 2005] and Roy Cook [Cook, 2010] to establish the range of the notion of pluralism. The forms of pluralism of J.C. Beall and Greg Restall [Beall and Restall, 2006], Stewart Shapiro [Shapiro, 2014], Restall by himself [Restall, 2014], Francesco Paoli [Paoli, 2014] and Ole Hjortland [Hjortland, 2013] are the main theories considered, along with the framework of B-entailment [Blasio, 2016a], which is based on a cognitive attitude semantics and for which a sequent calculus is presented.

B-entailment is a four-place consequence relation that generalizes the usual framework of logical consequence. The thesis proposes to analyze the framework of B-entailment as a useful ground for pluralism. B-entailment offers a framework through which different notions of consequence can be expressed and, as such, is interesting from the point of view of logical pluralism. This claim is further elaborated by explaining how this framework might fit into different varieties of pluralism presented.

Chapter One, "Remarks on Pluralism", clarifies the terminology to be employed and introduces the varieties of pluralism to be discussed. Chapter Two, "B-entailment Presented", introduces B-entailment and a semantics for it based on cognitive attitudes. It also explores a sequent calculus for its non-deterministic semantics. Finally, in Chapter Three, "Pluralism Revisited", B-entailment is examined in the light of the varieties of logical pluralism introduced in Chapter One by taking in consideration the formal developments of Chapter Two. This chapter argues for pluralism in the framework of B-entailment and that this framework is interesting for the pluralist because of its structure. In the Conclusion, the argument for pluralism within B-entailment is revisited and further work is proposed.

1 Remarks on Pluralism

This initial chapter deals mainly with the nature and range of logical pluralism. It also briefly lays out some vocabulary used in the course of the thesis. A short taxonomy of pluralism is presented, along with the pluralisms of Stewart Shapiro, J.C. Beall and Greg Restall, Greg Restall, and Ole Hjortland. Shapiro's eclectic approach will be adopted in this dissertation, so it makes sense to start there. The other varieties of pluralism are then discussed somewhat in light of Shapiro's analysis. Chapter 2, "B-entailment Presented", introduces B-entailment and a semantics for it based on cognitive attitudes. It also explores a sequent calculus for logics based on B-entailment and examines its non-deterministic semantics. Finally, in Chapter Three, "Pluralism Revisited", B-entailment is treated in the context of the varieties of logical pluralism introduced in Chapter One by taking into consideration the formal developments of Chapter Two. In the Conclusion, the argument for pluralism within B-entailment is revised, and the nature of logical consequence discussed.

1.1 Vocabulary

This section establishes some key terms used in the course of this thesis.

A formal language is formed by a set of atomic formulas and and a set of propositional connectives, with sentences of the language being recursively defined on the set of n-ary connectives by n formulas.

Definition 1. (Valuation) Given a language $\mathcal{S}$ and a set of truth values $\mathcal{V}$, a *valuation* is a function v from each sentence $\varphi \in \mathcal{S}$ to a set of truth values $x \in POW\{\mathcal{V}\}$.

Given an arbitrary language, a *logical consequence relation* or an *entailment relation* is a relation between two sets of formulas from that language (where capital greek letters are sets of formulas and small greek letters are formulas):

Definition 2. (Logical consequence relation) Given a set of truth values $\mathcal{V}$, let $\mathcal{D}$,$\mathcal{U} \subseteq \mathcal{V}$ and let Γ and Δ be sets of proposition from a given language. Then $\Gamma \vdash \Delta$ (Δ is a *logical consequence* of Γ) iff given a valuation v, for all $\gamma \in \Gamma$ and $\delta \in \Delta$, when $v(\gamma) \in \mathcal{D}$, then $v(\delta) \in \mathcal{U}$.

When the conclusion is a singleton, the consequence relation is called "single conclusion", when not, it's called "multiple conclusion".

An argument is a statement of the form $\Gamma \vdash \Phi$, where conclusion (Φ) is a logical consequence of the set of premisses (Γ), and validity is a property of an argument such that, when it is the case that the consequence relation holds between the premisses and the conclusion, then $\Gamma \vdash \Phi$ is said to be valid.

A logical consequence relation is "Tarskian" if it is reflexive, transitive and monotonic. Given two sets of sentences Γ and Δ if the intersection of Γ and Δ is not empty, then $\Gamma \vdash \Delta$ and $\vdash$ is said to be "reflexive". A consequence relation is "transitive" when given an argument which has as premisses that which another argument has as a conclusion, a third argument can be given straight from the premisses of the first argument to the conclusion of the second (if $\Gamma \vdash \varphi$ and $\varphi \vdash \Delta$, then $\Gamma \vdash \Delta$). A consequence relation is "monotonic" when once some conclusion is entailed by some premisses, the addition of other premisses and conclusions does not invalidate the argument (if $\Phi \vdash \Psi$ then $\Phi, \Gamma \vdash \Psi, \Delta$).

A *logic* is a tuple $\langle L, \vdash_1, ..., \vdash_n \rangle$, in which L is a formal language and each $\vdash_i$ is a logical consequence relation. This definition draws on Shramko and Wansing's idea of a multi-dimensional logic [Shramko and Wansing, 2011].

1.2 Pluralism aplenty

There are various formulations of logical pluralism, some more interesting than others. A not unreasonable approach to the topic is to define logical monism and to define logical pluralism as its negation. In short, logical monism is the belief that there is just one correct, best or legitimate logic [Cook, 2010, p. 492], and by contrast, logical pluralism is the view that there is more than one correct or legitimate logic. Unpacking this possibility leads to different kinds of pluralism.

Another approach is by way of relativism, which term itself means different things. Following Shapiro [Shapiro, 2014, p. 7], one may say that: "Y is relative to X", such that Y is the dependable variable and X the independent variable. The frame of reference X is prior to determining Y. For example, judgments of etiquette are relative in this way, because one must first fix conventional norms (X) before making judgments about violations of these (Y). Roy Cook asserts that "relativism about a particular phenomenon is surely the most straightforward route to pluralism" [Cook, 2010, p. 493], in the case that relative to the same X, more than one Y is accepted. At the same time, relativism does not imply pluralism nor the other way around, as it is possible to accept only one Y after X has been fixed (which would not be a pluralism) and it is also possible to be a pluralist without being a relativist (in the case of accepting more than one Y independently of any X).

A trivial form of pluralism is called by Cook "Mathematical Logical Pluralism", which is simply the claim that there is more than one logic (such as Classical Logic, multi-valued logics, and modal logics), and which is similar to a pluralism presented by Graham Priest in which "there are many pure logics" [Priest, 2005, p. 2]. This is analogous to a pluralism in geometry, where there are many pure geometries (such as Euclidean, Riemannian and spherical) and each is a "perfectly good mathematical structure" [Priest, 2005, p. 2] and there is no question of rivalry between geometries at this level, at least not anymore. There have been periods in the history of logic in which this claim was disputed, such as when Sylogistic Logic or Classical Logic were taken to be the one correct logic. Yet ever since the development of non-classical logics and their acceptance by the mainstream logic community, multiple logics have been accepted as valid structures. In the absence of applications, there would be no rivalries among logics and this form of pluralism is trivial, in the sense of being irrelevant to the contemporary context.

With fixed application in mind, more substantial form of pluralism arises, which Cook calls "Mathematical Application Pluralism". Priest also acknowledges a pluralism of this kind, namely, theoretical pluralism. He argues that when an application is fixed and a pure logic is applied by interpreting it in some way, the logic becomes a theory of how that domain is interpreted, and in this respect, "there may well be disputes about which theory is correct" [Priest, 2005, p. 3]. These disputes are adjudicated in logic "by the usual criteria of theoretical evaluation"[1] [Priest, 2005, p. 3]. For Priest, this form of pluralism is unjustified, because he believes one logic will come out better than another in a dispute between them for a specific application. Cook says that this pluralism "amounts to the claim that there are at least two distinct logics that can be fruitfully applied to represent aspects of the world" [Cook, 2010, p. 494], and one logic does not need to be better than the other. For example, different modal logics might be claimed to be the

[1] There criteria are, for instance, adequacy to the data, simplicity, consistency, power, avoidance of *ad hoc* elements. [Priest, 2015].

right one to talk about time. While Priest thinks there are criteria that can be used to decide which logic is best, Cook does not call for such decisions.

A third level of pluralism arises when Priest speaks of the "canonical application" of a pure logic, that is, its application to the analysis of reasoning. In this form of pluralism, there is a translation from the vernacular language to a formal language, and "a vernacular inference is valid iff its translation into the formal language is valid in the pure logic" [Priest, 2005, p. 4]. In a similar vein, Cook mentions a "Logical Consequence Pluralism", such that there is more than one logic, $< L, \Rightarrow >$, for which, there being a mapping from the formal language L into natural language, an argument is valid if and only if it is a logical consequence of $< L, \Rightarrow >$. In this case, there are many ways to make such translation, pluralism being simply having more than one language for which this translation works. Cook and Priest both deny that this is a serious form of pluralism, in as much as the dispute is among translations, not in the form of reasoning. For example, it's possible to translate "or" from a natural language as either an inclusive or an exclusive disjunction in a formal language, and this does not lead to a pluralism of logics directly, but merely of translations.

While Priest goes on to argue for monism with regard to the canonical application, Cook examines a pluralism such that when there is "one fixed formal language and logical/nonlogical divide, there is more than one logic that correctly codifies logical consequence in natural language" [Cook, 2010, p. 497], which would be having at least two logics, with different consequence relations, for the same formal language. Cook calls this "Substantial Logical Pluralism". He defines this as follows: given a mapping from the formal language L into the natural language, there are two distinct logics $< L, \Rightarrow_1 >$ and $< L, \Rightarrow_2 >$ such that $\Rightarrow_i$ are consequence relation. In the notation adopted here, this would be a pluralism of the form $< L, \Rightarrow_1, \Rightarrow_2 >$, once there be only one formal language and two consequence relations on the same language. A pluralism of this kind can happen between classical and intuitionist logic, such as done by Restall (section 1.5 bellow).

Finally, worth mentioning are two historical precursors to pluralism. Acording to Cook [Cook, 2010, p. 495], Alfred Tarski presents a view in which the choice of logical terms determines consequence and validity [Tarski, 1983]. The demarcation of logical and non-logical terminology provides for the possibility of different logics, and as such, is a small step towards pluralism. This step would be something along the lines of a Mathematical Logical Pluralism in Cook's terminology, which is the claim that there are different formal logics, and hence not taken as a serious form of pluralism.

Another precursor to logical pluralism, according to Cook [Cook, 2010, p. 497], is that of Rudolf Carnap [Carnap, 1937, p. 52] who discusses the notion of tolerance. In brief, one may to choose a framework or system of language which to work with as it suits one's needs, and questions regarding the correctness of a logic are relative to the chosen framework. In this way, different logics can be proposed for different languages and it makes no sense simply to ask which logic is the best. This view is not taken to be what is meant by pluralism in the current sense of the word [Restall, 2002]. Cook claims that Logical Consequence Pluralism holds for this view [Cook, 2010], because it amounts to saying that there is more than one language with which a logic can be defined.

Carnap's view gives rise to a debate about meaning variance and how it relates to the dispute among logics. Carnap's idea is that "any difference in logical consequence is due to a difference in languages" [Restall, 2002, p. 4]. What one wishes for in a pluralism these days is that accepting two different logics does not come accompanied by having to accept two different languages. W.V.O. Quine would say that both parties are just talking past each other with no real dispute [Shapiro, 2014, p. 120] but only meaning variance. For there to be a real dispute, the language would need to be the same for different logics. Carnap's pluralism is only a precursor to pluralism in the contemporary setting, as his pluralism arises from a difference of languages. By dint of this debate, meaning variance is usually taken to be a problem in a pluralism, for it indicates the pluralism not to be a proper pluralism. The varieties of pluralism presented in this chapter each propose a solution to this problem, and intend to formulate a pluralism in which there is more than one consequence relation on the same formal language.

In conclusion, a theory of logical pluralism must avoid the meaning variance problem, and to do so, needs to fix a formal language on which to define at least two different consequence relations.The remainder of this chapter examines some varieties of logical pluralism which seek to be of this kind, as well as Shapiro's pluralism, which goes beyond this formulation and to discuss the issue of meaning variance in more detail.

1.3 Eclectic pluralism

In his *Varieties of Logic* [Shapiro, 2014], Shapiro sets out to explore many ways in which one can be a relativist or pluralist about logic. With this main theme in mind, many other adjacent themes are explored, such as that of the legitimacy of using more than one logic in a mathematical theory, the nature of validity and logical consequence, meaning shifts, and foundational studies in mathematics. Throughout the book, Shapiro adopts what he calls an eclectic aproach to logic, and this pluralism is called "eclectic pluralism" [Geerdink and Novaes, 2016].

In Chapter One, Shapiro begins with the concept of folk-relativism, which is attributed to Crispin Wright, and for which the slogan is, "There is no such thing as simply being Φ". Chris Swoyer proposes a "general relativistic schema", to wit, Y is relative to X. In this schema, Y is the dependable variable and X is the independent variable (a frame of reference). For example, moral principles might be said to be relative to a culture. In keeping with the issue of the book, logical consequence and validity are taken to be the dependable variable Y, which are relative to a logic or formal system/semantics. So to answer the question "Is this argument valid?", one must first fix a logic, and the validity of the argument will be relative to this choice.

Shapiro then presents several options for interpreting folk-relative expressions, attributed to John MacFarlane. These views are used throughout the book to address issues of meaning. The first one, called "indexical contextualism", "is the view that the content expressed by the term is different in different contexts of use" [Shapiro, 2014, p. 10]. Examples of words apt for this kind of treatment are "enemy", "left", "right", "ready", and "local". The second one, called "non-indexical contextualism" is the view that the content of the term expressed "does not vary from one context of use to another, but the extension of the term can so vary" [Shapiro, 2014, p. 11]. For example, Person 1 might say, "Carrot cake is tasty", and Person 2 might say, "Carrot cake is not tasty". For each, "tasty" means the same, but the extension of this predicate for each is different. Finally, "assessment-sensitive relativism" is the view in which "the content of the term does not vary from one context of use to another" [Shapiro, 2014, p. 12], and also "the term gets its extension from a possibly separate context of assessment" [Shapiro, 2014, p. 12]. A Person 3 might show up and agree with Person 1, that the cake is delicious, because from Person 3's context of assessment, Person 1 is correct. Shapiro notes as well that "the difference between non-indexical contextualism and assessment-sensitive relativism is made manifest by the phenomenon of retraction" [Shapiro, 2014, p. 13]. In the above example, if at a later date Person 3 comes to find that carrot cake is not tasty, then she should retract the utterance, because "from her later context, where the assessment is taking place, [the] earlier utterance — the thing she said in the early context of use — is false" [Shapiro, 2014, p. 13]. In the case of both kinds of contextualisms, "there should be no retraction of the earlier utterance, as it remains true in its context of use" [Shapiro, 2014, p. 13].

After this explanation of kinds of relativism, pluralism is taken to be the view in which "different accounts of the subject are equally correct, or equally good, or equally legitimate, or perhaps even true" [Shapiro, 2014, p. 13]. This is different than folk-relativism because here folk-relativism is simply the claim that there is more to consequence relation than simply being true. For example, to the question "Is this argument valid?", a folk-relativist would reply that the answer depends on the logic. Folk-relativism thus gives rise to pluralism, as for the pluralist the answer to this question is not only that it depends on the logic, but also that more than one answers is equally correct.

In Chapter Two, Shapiro articulates four ways in which someone can be a folk-relativist or pluralist concerning logical consequence. A brief summary of each path to pluralism follows.

The *first* option is that there is no one single notion of "validity" and of "logical consequence". Shapiro points out "that logical consequence is either polysemous or is something like a cluster concept" [Shapiro, 2014, p. 17]. In this pluralism, there are several articulations of logical consequence (the author enumerates nine of them) and each one is an equally good account of the notion in question. For example, the logical consequence of $\Gamma \vdash \varphi$ can be thought of in terms of possible worlds, where there is no world in which φ holds and every member of Γ does not hold; in terms of truth, where it is not possible for every member of Γ to be true and φ false, or also in terms of the language, where φ holds in every interpretation of the language in which every member of Γ holds.

A *second* pluralism presented deals with the perspective of "logic-as-model". In this approach, pluralism emerges from logical systems being mathematical models that interpret sentences. "With models generally, there are usually tradeoffs, depending on the underlying purpose of the model and what, exactly, is being modeled" [Shapiro, 2014, p. 206]. Usually there is not just one best model for anything, especially because the polysemous notion of logical consequence. Beall and Restall's pluralism is of this kind.

Thirdly, a pluralism based on the distinction between logical and non-logical terminology is presented. Is there only one correct demarcation of what is logical? Shapiro argues that there is not, and so "logical consequence and validity are relative to a choice of logical terms" [Shapiro, 2014, p. 206]. This posture gives rise minimally to a folk-relativism, where the question of whether an argument is valid or not depends on a choice of logical terminology.

Fourth and finally, the author argues that the notion of logical consequence is vague. This stance depends on the account of vagueness and how vague expressions are seen as relative to something, such as a sharpening or a conversational context, which leads minimally to a folk-relativism. This vagueness is within one notion of consequence relation, differently than the first option presented above, in which there are more than one notion.

In Chapter Three, three intuitionistic mathematical theories are presented: Heyting arithmetic + Church's thesis; intuitionistic analysis; and Kock-Lawrence's smooth infinitesimal analysis. It is held that even these theories which disagree with classical logic are interesting, fruitful (or useful), and legitimate branches of mathematics. These examples set forth a pluralism, or folk-relativism, concerning logic. The bottom line of this chapter is that consistency is all that matters for legitimacy (view attributed to David Hilbert and called the "Hilbertian view"). This approach is what Shapiro calls an eclectic approach to logic.

Shapiro entertains the possibility of adopting a weaker standard for legitimacy and of changing the Hilbertian slogan of "consistency implies existence" to "non-triviality implies existence" and thus be able to acomodate paraconsistent logics. Would this be an interesting move? "[A]re there any interesting and/or fruitful inconsistent mathematical theories, invoking paraconsistent logics?" [Shapiro, 2014, p. 84]. He notes that there are people working on this and, since it might come about that useful applications will be found for these logical theories, this possibility should not be dismissed.

In Chapter Four, Shapiro deals with the issue of meaning of logical terms. Two batches of questions are presented. The first batch can be roughly stated as, "do 'valid', 'logically entails' and 'logical consequence' have the same meaning in different theories?" and the second as, "do 'and', 'or', 'if-then', 'for all' and 'there is' mean the same in different theories?" For the first questions, Shapiro says that whoever introduces the terms can stipulate their meaning, so "valid" and "logical consequence" are taken to be relative to a structure, which leads to a folk-relativism concerning logic. With regard to the second batch of questions, the meanings of the logical terms are stipulated as the system is constructed, by giving satisfaction conditions and/or derivation conditions. The remainder of the Fourth Chapter investigates two answers to this last question – do logical particles have the same meaning in various theories or different meanings? – and places this discussion within MacFarlane's taxonomy of relativism[2].

For the first option, Shapiro supposes that logical particles have the same meaning in different theories, even if the logics are different. There are things which are valid in one theory and invalid in another, so there is, for example, valid-in-classical-theories and valid-in-intuitionistic-theories. Under this supposition, there is at least folk-relativism about "valid" and "logical consequence" and possibly pluralism. Where does this fit within the schema of relativism presented in a previous chapter?

> The classical logician says that classical reductio is valid; the intuitionist says that classical reductio is invalid. By the assumption of this section, they are talking about one and the same argument form. By the perspective of this study, both are correct. That is, both utterances are true, in their respective contexts. In line with the contextualism, they do not really disagree, since "valid" has a different content in their respective utterances. [Shapiro, 2014, p. 89]

This first option then, in which the logical particles (connectives and quantifiers, for example) have the same meaning, is not a case of indexical contextualism, because this would lead the particles to have different meaning in each context of use, contrary to the supposition. Furthermore, if one accepts Shapiro's eclectic orientation towards logic,

2 Indexical contextualism is the view in which the content of the term depends on the context of use, while non-indexical contextualism is the view in which the content does not depends on the context, instead, the extension of the term varies according to context. Assessment-sensitive relativism, moreover, is the view in which the content also does not depends on the context, but the term gets its extension from a possibly separate context of assessment.

assessment-sensitive relativism in an unrestricted form does not make much sense because there is not a single context of assessment. Shapiro concludes that assuming "that the logical terminology has the same meaning in all theories, no matter what the logic, the proper conclusion concerning the logical terms is non-indexical contextualism" [Shapiro, 2014, p. 120]. That is to say, the meaning of the terms does not vary, but the extension of them do, so that "the terms have the same content but get evaluated differently in the different theories/contexts" [Shapiro, 2014, p. 120]. For instance, negation in classical and in intuitionist logic is the same thing, but get evaluated differently in each theory. In this scenario, then, "validity" and "logical consequence" have different contents when used in the different theories (as these are the extension in which the logical particles get evaluated), and this could give rise to pluralism if one accepts more than one account as correct.

For the other option, suppose instead that the connectives and quantifiers have different meanings in different logical systems and structures. Thus the meanings of logical terms are different in each context and in this case there is an indexical contextualism. Shapiro argues that if the connectives and quantifiers have different meanings, this gives rise to a monism regarding logical consequence, where each choice of logical connectives leads to a different logic. This choice of meaning for the connectives is what Cook calls Logical Consequence Pluralism (in section 1.2 above), where different translations lead to different logics but there is no pluralism because the dispute is between translations, not between notions of consequence.

Shapiro does not choose between taking particles to have different meanings in different logical theories or not. Yet later in Chapter Seven, however he suggests that having the same meaning of logical particles could be the more interesting choice, because it provides for a way to compare different theories. The expression "same meaning as" is clarified as well, indicating that it is also context-sensitive.

Taking his analysis of meaning variance a step further in the Fifth Chapter, Shapiro explores the issue of meaning shift, that is, whether the word "not" (for example) has the same meaning for a classicist or intuitionist and a relevantist. The purpose of the chapter is to show that the meaning of logical terminology is itself context-sensitive. In some conversational context, it makes sense to say the connectives of different logics have different meanings, and in other conversational contexts, it makes sense to say the terminology is the same in different logics, because the relation "same meaning as" is vague.

Although there is no consensus on what makes a relation vague, vagueness is commonly linked to a sorities series. This move, however, should be avoided. What makes for vagueness is in part the presence of borderline cases. Shapiro points out that vague terms, specially gradable adjectives, do exhibit some context-sensitivity, but it is an open question how this should be articulated. He embraces Delia Graph's view that "the extensions of vague terms vary according to the interests of speakers at a given moment" [Shapiro, 2014, p. 102] and also Scott Soames's view that "vague terms are indexical expressions whose contents, and thus, extensions vary from context of utterance to context of utterance" [Shapiro, 2014, p. 102]. Shapiro and Diana Raffman defend similarly a position in which "competent speakers of the language can go either way in the borderline regions of a predicate or relation, at least sometimes, without compromising their competence in the use of these words" [Shapiro, 2014, p. 102]. These contextualist views on vagueness, plus the eclectic approach put forth so far, predict that in some situations the logical particles have the same meaning, and also that in some situations in which there is meaning-shift, the "meaning-shift of logical particles is itself context-sensitive" [Shapiro, 2014, p. 103].

Furthermore, the question of meaning shift presupposes an analytic-synthetic distinction. Here Shapiro embraces Friedrich Waismann's notion of open-texture.

> Let P be a predicate from natural language. According to Waismann, P exhibits open-texture if it is possible for there to be an object a such that nothing concerning the established use of P, nor the non-linguistic facts, determines that P holds of a, nor does anything determine that P fails to hold of a. [Shapiro, 2014, p. 142]

Empirical concepts do not delimit all possible directions, for an unforeseen circumstance might always arrive which causes one to have to modify the definition. Concepts are open-textured in so far as one cannot define a concept with absolute precision. Waismann finds terms and expressions like "analytic", "synonymous", and "means the same as" to be subject to open-texture. This claim is also what Shapiro wants to maintain regarding logical particles in various mathematical theories. That is not to say that these expressions are trivial, but rather that their meaning is not fixed, being relative in some way.

In Chapter Six, Shapiro deals with the logic of foundational studies, asking what is the proper logic of a foundational study and if there is more than one such logic, for more than one such foundation. This approach, which extends the Hilbertian view to logic, must determine the proper logic for application in foundational studies. It is observed by Shapiro that this is the logic of meta-language, in opposition to that of an object-language.

In the area of foundational studies, the most famous theory for this purpose is set theory. Shapiro atributes de following view to Penelope Maddy.

> Set theory hopes to provide a dependable and perspicuous mathematical theory that is ample enough to include (surrogates for) all the objects of classical mathematics and strong enough to imply all the classical theorems about them. [Shapiro, 2014, p. 129]

Zermelo-Fraenkel Set Theory does the job well enough for classical mathematics, but not so much when the underlying logic of a theory is not classical. What would be the logic of the foundation of all mathematical theories, classical and non-classical?

It is argued that there cannot be such single arenas, or at least not a "single domain" in the most obvious way, and so "we cannot think of our grand foundation as a theory about (surrogates for) all mathematical objects, construed as a single domain with a single language" [Shapiro, 2014, p. 175]. Category theory is an alternative to set theory, but it is also based on classical logic. Within category theory, there is topos theory, with at least an intuitionistic "internal logic".

Shapiro embraces a pluralism of foundational studies [Shapiro, 2014, p. 181], accepting more than one foundation as correct at the same time. In this multi-foundational setting, there would be interest in the relationship between these various foundational studies. Each foundational study could have an account of the relationship between itself and other foundational theories. Set-theory claims to have a grasp of category theory, whereas category theory regards set theory within its purview. There is no need to have a single foundation encompassing every possible mathematical theory. As Shapiro asks, "[H]ow would we know that we had such a theory, unless we had some reason to think we know what all of the possibilities are?" [Shapiro, 2014, p. 181] .

Chapter Seven treats of what does and does not follows from a fixed logic, focusing on how a meta-theory evaluates an object-theory. Considering that there is no "God's eye view", the question of how to evaluate a logic from the perspective of another logic arises. There is no need for the meta-logic to be the same as the object-logic. Shapiro asks about the appropriate logic to evaluate statements of the form "γ follows from Γ in logic L". Is there only one? Shapiro notes that sometimes the logic chosen to evaluate such statements has some bearing on the truth-value of statements of this form and sometimes not.

The book concludes with remarks on how, in the eclectic orientation presented, logicians are free to study logics different than their own, learn how these logics relate to each other, and on how a third logician can use yet another logic to study the previous meta-theoretic projects and how these logics relate to each other.

Varieties of Logic starts out with an open-mind with regard to what counts as logic (the Hilbertian view) and goes further to elaborate this open-mindedness to issues of meaning of logical terms and meta-theories. Shapiro's pluralism detaches itself from the supremacy of classical logic, leaving open the possibility for non-classical logics to bloom and become applicable in mathematics and for meta-theoretical purposes. This pluralism is more methodological than the ones which will be explored ahead, as it leaves open the possibility of exploring logic in a setting that is tolerant different approaches to it. In Shapiro's words, "In the intellectual world, new mathematical theories crop up all of the time, and legislating for the future, from a philosophical armchair, is risky" [Shapiro, 2014, p. 95].

1.4 GTT pluralism

J.C. Beall and Greg Restall have put forth their own version of logical pluralism [Beall and Restall, 2006]. For them, pluralism is taken to mean that more than one logic is correct for a certain language and for certain connectives of that language. Their form of pluralism is based on the vagueness of the notion of "following from". They propose different ways in which one can settle the account of deductive logical consequence (in opposition to inductive consequence). Each such way is called a "precisification", which is one way to settle the notion of logical consequence. A logical consequence relation for this purpose must be necessary (in some sense), formal (schematic or topic-neutral), and normative ("there is a clear sense in which someone who reasons according to patterns which it rules invalid makes a mistake" [Burgess, 2010, p. 520]), as well as Tarskian.

Beall and Restall define their form of pluralism from the so-called "Generalized Tarski Thesis" (GTT): "An argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion" [Beall and Restall, 2006, p. 29]. Pluralism arises when one accepts that there is more than one precisification that can stand for x. By validity, the authors mean preservation of truth in all cases, which are dependent on the precisification at hand.

Beall and Restall mainly discuss three logics (in the sense of precisifications): classical, intuitionistic and relevant. For classical logic, the cases in question are Tarskian models. The truth conditions of the connectives are defined in the usual way, as being true in a model. For intuitionistic logic, the cases are constructions (which can be incomplete), and the connectives are defined for constructions. For relevant logic (none are in particular specified), the cases are situations, which can be incomplete and inconsistent, and the connectives are defined for situations. Logics must be reflexive, transitive and monotonic.

Beall and Restall's take on pluralism has been faulted from several angles. Manuel Bremer [Bremer, 2014] and Rossana Keefe [Keefe, 2014] argue separately to the effect that this pluralism is biased, for though Beall and Restall consider different logics, they seem to take classical logic to be better than the others. Rosanna Keefe points out that intuitionistic and relevant logic are treated by Beall and Restall as "nothing more than occasional restrictions of the one true logic" [Keefe, 2014, p. 295], and Manuel Bremer calls this "the superior judge problem". Another argument against this form of pluralism, given by Priest [Priest, 2005], doubts Beall and Restall's pluralism is a pluralism at all. Priest argues that the change in truth-conditions for the connectives causes a change in meaning of the connectives, therefore there are simply different competing logics for the same application, which for him is not a case of pluralism. Other questions are raised as to exactly what counts as a case [Goddu, 2002].

Shapiro discusses this pluralism and says of it that it is interesting because it argues for a pluralism within the model-theoretic approach. It is, however, very narrow insofar as it only counts as logic what can be captured by GTT. It leaves out interesting logics such as those that are not transitive and not reflexive and it also ignores the proof-theoretic tradition. Shapiro also mentions some of the criticism this pluralism has received, such as by Priest and Goddu.

If one does not take into account the criticism it has received this pluralism fits into Cook's Substantial Logical Pluralism, because there are two distinct logics, such as classical and intuitionistic, that encode natural language. It fails to be a significant pluralism by Priest's account, as he argues that this theory does not offer a pluralism at all, only one logic with classical or intuitionist connectives. If this criticism is accepted, then it is only a kind of Mathematical Application Pluralism, as the pluralism arises from the multiplicity of languages and not of consequence relations.

1.5 Proof-theoretical pluralism

Restall later examines logical pluralism in terms of different standards of proof, as to understand another way in which there can be different consequence relations in one language [Restall, 2014]. This posterior work is taken to be a proof-theoretical basis for the previous work with Beall, which relies on semantics. Restall begins talking of different kinds of validity. If one asks if an argument is valid, the answer can be two-fold:

> [Y]es, the argument is valid—in the sense that it is impossible for the premises to be true and the conclusion to be false—and no, the argument is invalid, for there are impossible circumstances (circumstances inconsistent about p, but in which q fails) in which the premises hold and the conclusion doesn't. The one and the same argument can be both valid in one sense and invalid in the other. [Restall, 2014, p. 280]

There are, to the point, two kinds of validity, classical and constructive. An argument can be classically valid, but fail to be constructively valid, such as the inference from $\neg\neg p$ to p. In such cases, it is not the meaning of negation that changes, but the notion of consequence. In the classical consequence, for instance, inconsistent premises entail every conclusion.

Restall remarks that although in *Logical Pluralism* [Beall and Restall, 2006] the picture of pluralism presented relies on a model-theoretic truth-preservation account of logical consequence, "we can understand the difference between classical and intuitionistic logics in terms of proof" [Restall, 2014, p. 228].

Restall considers Gentzen's sequent calculus for classical logic, supplemented by the structural rules for Identity and Cut. The rules for negation, conjunction and disjunction are:

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\,[\neg L] \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\,[\neg R]$$

$$\frac{X, A \vdash Y}{X, A \wedge B \vdash Y}\,[\wedge L_1] \qquad \frac{X, B \vdash Y}{X, A \wedge B \vdash Y}\,[\wedge L_2]$$

$$\frac{X \vdash A, Y}{X \vdash A \vee B, Y}\,[\vee R_1] \qquad \frac{X \vdash B, Y}{X \vdash A \vee B, Y}\,[\vee R_2]$$

If one considers only proofs in which there is at most one formula on the right hand side of a sequent, then these are intuitionist instances of the rules for the connectives:

$$\frac{X \vdash A}{X, \neg A \vdash}\,[\neg L] \qquad \frac{X, A \vdash}{X \vdash \neg A}\,[\neg R]$$

$$\frac{X, A \vdash C}{X, A \wedge B \vdash C}\,[\wedge L]_1 \qquad \frac{X, B \vdash C}{X, A \wedge B \vdash C}\,[\wedge L_2]$$

$$\frac{X \vdash A}{X \vdash A \vee B}\,[\vee R_1] \qquad \frac{X \vdash B}{X \vdash A \vee B}\,[\vee R_2]$$

If one considers only proofs in which there is at most one formula on the left side of a sequent, then these are dual-intuitionist[3] intances of the rules for the connectives:

$$\frac{\vdash A, Y}{\neg A \vdash Y}\,[\neg L] \qquad \frac{A \vdash Y}{\vdash \neg A, Y}\,[\neg R]$$

$$\frac{A \vdash Y}{A \wedge B \vdash Y}\,[\wedge L]_1 \qquad \frac{B \vdash Y}{A \wedge B \vdash \neg Y}\,[\wedge L_2]$$

$$\frac{C \vdash A, Y}{C \vdash A \vee B, Y}\,[\vee R_1] \qquad \frac{XC \vdash B, Y}{C \vdash A \vee B, Y}\,[\vee R_2]$$

So there are classical derivations, which are trees of sequents, whose leaves are axioms and whose transitions are instances of the structural rules or the rules for the connectives,

[3] This logic is the semantic dual of intuitionist logic.

intuitionist derivations, which are classical derivations with the constraint of at most one formula on the right hand side, and dual-intuitionist derivations, which are classical derivations with the constraint of at most one formula on the left hand side. A derivation then can have different virtues: that of being classically valid, or of being intuitionistically valid or of being dual-intuitionistically valid. Some derivations have all three virtues. The same statement, $p \vee \neg p$ for example, is both a classical and a dual-intuitionist tautology, but it is not an intuitionist one. "On this picture, we have one language, and three logical consequence relations on that language, by having one family of derivations and various different criteria which may be applied to those derivations" [Restall, 2014, p. 284].

Restall proposes to take the proof-theoretic account of validity as primary, and "soundness and completeness for the model theory [as] a criterion for the acceptability for that model-theoretic analysis" [Restall, 2014, p. 285]. In this point, this work by Restall tries to justify the work done together with Beall [Beall and Restall, 2006] by giving a counterpart to the semantic approach to logical pluralism presented there.

Next Restall argues that it is not the case there are "three negations in one logic", which would be the case if one were thinking model-theoretically. He characterizes the truth-conditions for classical, intuitionist and dual-intuitionist negation, and argues that these three negations cannot co-inhabit a logic. He points out that "there is no frame in which all three negations coexist as propositional operators on the same class of propositions, giving the distinct classical, intuitionist and dual-intuitionist properties" [Restall, 2014, p. 287]. Due to its heredity condition[4], intuitionist negation and classical negation cannot coexist, as this would cause the heredity rule to be broken[5].

It is possible to have intuitionistic and dual-intuitionistic negation without breaking the heredity rule, and "on this picture, there's one logic, encompassing statements expressible in two different vocabularies" [Restall, 2014, p. 287].

The connection to model theory is made though point of a model, which are a place for a counterexample of an invalid argument. If B cannot be derived from A, then there is a point in some model in which A holds and B fails. So a point in a model is an idealized

4 The "heredity condition" in Intuitionistic Logic is the notion that once one has some information about φ at some point in a frame, on the next point that information remains. That is, given two points $c \leq d$ in a frame, d is at least as consistent as c.

5 Given $c \leq d$, it is possible for $\neg p$ to be true at c and $\neg p$ to be false at d, so even though d extends c, there are more things true at c than at d.

underiverable sequent. There are different ways in which a sequent may be invalid, so there are also different kinds of points. In his terminology, worlds are called classical points, constructions are intuitionist points, and situations are dual intuitionist points. The truth conditions for the connectives are given in terms of worlds, constructions and situations, which, for example, satisfy the rules for classic, intuitionist and dual intuitionist negation, respectively. In this view, there is only one negation, and the variation in clauses don't change the negation at each point, but change the points themselves. "Negation is a single item, and the difference between worlds, constructions and situations governs the different behaviour of negation at those three different sites" [Restall, 2014, p. 290].

Hjortland sumarizes this view of proof theoretic plurality of sequent-calculus as: "one language, one proof system, but three provability relations" [Hjortland, 2013, p. 12]. Hjortland, however, also criticises this view [Hjortland, 2014], which he characterizes as a minimalism for logical connectives. In this view, minimal conditions are established for the logical connectives with different consequence relations being established by the different structural rules imposed on the system. Hjortland argues that this still leads to meaning variance, not of the logical connectives, but of the notion of validity. Turning attention from the connectives of the object language to the structural connectives of the sequent calculus framework, the comma and the sequent arrow, it can be claimed that different theories " will assign different meanings to the sequent arrow $\Rightarrow$ insofar as they yield different logics" [Hjortland, 2014, p. 15], targeting the notion of "'follows from'. This point is important because "if the concept of validity it not stable across the rival theories, there is no reason to think that the same subject matter is under dispute" [Hjortland, 2014, p. 15]. There would then be meaning-constitutive rules for the notion of validity, such at the cut-rule or reflexivity, which then makes the notion of consequence susceptible to meaning variance.

Another proponent of a pluralism similar to Restall's is Francesco Paoli [Paoli, 2003] [Paoli, 2014], who also argues for a semantic minimalism and faces Hjortland's criticism more directly. Paoli bases his pluralism in proof-theoretic semantics, which is based on Gentzen's senquent calculi. In this approach, it's important to build the semantics only in terms of the rules of the connectives, without reference to external models [Paoli, 2003, p. 336] and use sequent calculus as a framework for proof-theoretic semantics.

In this framework, there is a difference between operational and global meaning.

Comprehension of the first amounts to knowing the inferential practice of a connective c, as specified by the appropriate operational (introduction) rules for a connective. Comprehension of the second amounts to knowing the provable sequents which contain c. Paoli claims that most sequent calculi for available "deviant" logics share with classical logic the same operational rules for negation. Thus the operational meaning of negation is the same across a wide range of calculi. But all of these calculi assign a different global meaning to negation. Paoli says that this shows that "that there can be genuine conflict between logics without meaning variance" [Paoli, 2003, p. 539] because homophonic translation preserves the operational meaning of negation because they all operate as negation.

Paoli then proceeds to present this idea more formally. A presentation of a logic is a pair $\mathbf{L} = \langle L, S\rangle$, where L is a propositional language containing denumerably many variables and the connectives $c_1, ..., c_k$, and S is a cut-free sequent calculus based on L. The similarity type of a presentation $\mathbf{L}$, whose language contains the connectives $c_1, ..., c_k$, is the sequence of nonnegative integers $\langle n_1, ..., n_k\rangle$, where for each $i \leq k$ the number n_i is the arity of c_i. Two presentations are similar if and only if they have the same similarity type.

Another important notion if that of homophonic translation between two logics. If $\mathbf{L} = \langle L, S\rangle$ and $\mathbf{L}' = \langle L', S'\rangle$ are similar presentations and $\Gamma \Rightarrow \Delta$ is a sequent of $\mathbf{L}$ the homophonic translation $t(\Gamma \Rightarrow \Delta)$ of $\Gamma \Rightarrow \Delta$ into L' is defined as $(r, n, m \geq 0)$:

$t(p_4) = p_i$, for every variable p_i;
$t(c_i(A_1, ..., A_r)) = c_i(t(A_1), ..., t(A_r))$, for every connective c_i of L;
$t(A_1, ..., A_n \Rightarrow B_1, ..., B_m) = t(A_1), ..., t(A_n) \Rightarrow t(B_1), ..., t(B_m)$.
Based on these concepts, the genuine rivalry criteria for logics such that:

> (CGR) Let $\mathbf{L} = \langle L, S\rangle$ and $\mathbf{L}' = \langle L', S'\rangle$ be similar presentations, let $c_1, ..., c_k$ be the connectives in L and $c'_1, ..., c'_k$ be the connectives in L'. $\mathbf{L}$ and $\mathbf{L}'$ are genuine rivals iff, for every $i \leq k$, c_i has the same operational meaning (i.e. the same introduction rules) as c'_i but for at least an $i \leq k$, c_i and c'_i have different contextual meanings (i.e. either there is an S-provable sequent containing c_i whose homophonic translation into L' is not S-provable, or there is an S'-provable sequent containing c_i whose homophonic translation into L is not S-provable). If two logics have genuinely rival presentations, they are genuine

rivals. [Paoli, 2003, p. 540]

Three examples are analyzed under this criteria. Paoli agrees with Susan Haack that Modal Logics are not rival to Classical Logic, but only supplements it, because “CGR does not contradict this plausible assumption, since classical logic and any given standard modal logic have no similar presentations” [Paoli, 2003, p. 540]. Classical Logic is also not a rival to Quine’s fictional logic, because “ the homophonic translation is not acceptable in this case, for it does not preserve the operational meanings of the connectives in question” [Paoli, 2003, p. 540]. As to paraconsistent logic, there is at least one, subexponential linear logic without additive constants (LL) which is a genuine rival to classical logic, “because the latter and LL have similar presentations with the same stock of operational rules, yet there are classically provable sequents whose homophonic translations are not available in LL” [Paoli, 2003, p. 541].

Paoli’s pluralism can then be summed up through the acceptance of two rival logics, according to the CGR presented above. That is, there being two logics with the same connectives (through a translation) and same rules for them, which have different structural rules, thus they have two disinguished consequence relation for the same connectives.

The difference between Restall’s variety of pluralism and Paoli’s is that while the former uses sequent calculus to defend a model-theoretic pluralism, the latter relies only on sequent calculus, translations and a proof-theoretic semantics in terms only of rules with no reference to models. Both varieties are listed here together because both rely on structural rules to present the difference between consequence relations.

The issue of meaning variance in this kind of pluralism would not arise for Shapiro, who argues that meaning-variance is itself context sensitive. Of Restall’s approach, Shapiro comments that, as a framework for pluralism, it is limited to a proof-theoretic account and it is not broad enough for his purposes. Restall’s pluralism fits into Cook’s Substantial Logical Pluralism, but it is not an interesting pluralism for Priest, because it does not deal with the canonical application, that is, the analysis of reasoning.

1.6 Intra-theoretical pluralism

The pluralism proposed by Hjortland [Hjortland, 2013] is motivated by a critique of the pluralism of Beall and Restall in *Logical Pluralism* [Beall and Restall, 2006], which, as mentioned above, is accused of meaning-variance of the connectives. Hjortland proposes a *intra-theoretic pluralism* to avoid this problem by exploring a pluralism within one logical system (and one language) with multiple logical consequence relations.

Similarly to Shapiro, meaning-variance for Hjortland is defined by one of two theses: (A) the meaning of "valid" varies between logical theories, or (B) the meaning of some logical connective $\otimes$ varies between logical theories. Beall and Restall claim their pluralism subscribes to (A)-style meaning variance, but not to (B). Hjortland, however, argues otherwise, claiming that their pluralism of (A)-style meaning variance leads to a (B)-style one as well. On this basis, he holds Beall and Restall's pluralism to be inadequate, and puts forth his own pluralism, in which (A)-style meaning variance does not imply a (B)-style one.

Beall and Restall maintain their pluralism arises within one language (as they deny their theory is subject to (B)-style meaning variance) and the notion of "follows from" is vague and is subject to a precisification. Their view on logical pluralism (called by Hjortland *GTT Pluralism*) is that there are at least two different instances of GTT that are admissible precisifications of logical consequence. They subscribe to (A)-style meaning variance because of the unsettled nature of logical consequence. Hjortland argues that *GTT Pluralism* implies (B)-style meaning variance.

Since the meanings of the logical connectives are specified truth-conditionally, "it is natural to assume that the corresponding truth-conditions are given by the class of cases in question" [Hjortland, 2013, p. 7]. Thus for classical logic there are truth-in-a-model-conditions and for relevant logic there are truth-in-a-situation-conditions. The argument of Priest against Beall and Restall's pluralism is then plausible. Priest argues that the change of truth-conditions entails a change of meaning: "[i]f we give different truth conditions for the connectives, we are giving the formal connectives different meanings" [Priest, 2005, p. 204]. Hjortland concludes that "meaning-variance at the level of the consequence relation has carried over to the logical connectives" [Hjortland, 2013, p. 8].

He goes on to argue that it is possible to have (A)-style meaning variance without it leading to a (B)-style one. This would be a pluralism of logical consequence relations within one and the same theory, which he calls intra-theoretical pluralism. Hjortland says this view owes a lot to Restall [Restall, 2014]. This single logical theory could be one proof-system, or one formal semantics.

Hjortland claims that it is possible to have two logics which share the same interpretation of the logical connectives, but which treat the truth values differently, having thus two different consequence relation. For example, two logics are presented: Kleene 3-valued logic, K3, and Priest's LP. For both logics the truth values are $\mathcal{V} = \{0, i, 1\}$, while in regards to the designated value, K3 has $\mathcal{D} = \{1\}$ and LP has $\mathcal{D} = \{1, i\}$. Both logics share the same truth tables, with the consequence relation for each as such: an argument is valid ($\Gamma \vDash_{K3} A$) just in case, for every valuation v, whenever $v(B) = 1$, for every $B \in \Gamma$, $v(A) = 1$; and an argument is valid ($\Gamma \vDash_{LP} A$) just in case, for every valuation v, whenever $v(B) \in \{1, i\}$, for every $B \in \Gamma$, $v(A) \in \{1, i\}$. It is known that in K3 the Law of Excluded Middle is invalid, while in LP it is valid, and *ex falso quodlibet* does hold in K3 but fails to hold in LP, for "[i]n short, the two logics have distinct consequence relations. Even though they share the same interpretations of the connectives, they treat the truth-values differently with respect to validity" [Hjortland, 2013, p. 13].

Following this discussion, a 3-sided sequent calculus is presented both for K3 and LP in order to show that this pluralism also arises in the proof theoretic approach, in which there is a single proof system with more than one notion of provability. There are three initial sequents, which are generalization of the identity axiom of the form $A|...|A$, three forms of weakening and three forms of cut:

$$\frac{\Gamma_0|\Gamma_i|\Gamma_1}{\Gamma_0, A|\Gamma_i|\Gamma_1}\,(K_0) \qquad \frac{\Gamma_0|\Gamma_i|\Gamma_1}{\Gamma_0|\Gamma_i, A|\Gamma_1}\,(K_i) \qquad \frac{\Gamma_0|\Gamma_i|\Gamma_1}{\Gamma_0|\Gamma_i|\Gamma_1, A}\,(K_1)$$

$$\frac{\Gamma_0, A|\Gamma_i|\Gamma_1 \qquad \Gamma_0|\Gamma_i|\Gamma_1, A}{\Gamma_0|\Gamma_i|\Gamma_1}\,(Cut_{0,1}) \qquad \frac{\Gamma_0, A|\Gamma_i|\Gamma_1 \qquad \Gamma_0|\Gamma_i, A|\Gamma_1}{\Gamma_0|\Gamma_i|\Gamma_1}\,(Cut_{0,i})$$

$$\frac{\Gamma_0|\Gamma_i, A|\Gamma_1 \qquad \Gamma_0|\Gamma_i|\Gamma_1, A}{\Gamma_0|\Gamma_i|\Gamma_1}\,(Cut_{i,1})$$

The rules for disjunction and negation are given:

$$\frac{\Gamma_0, A|\Gamma_i|\Gamma_1 \qquad \Gamma_0, B|\Gamma_i|\Gamma_1}{\Gamma_0, A \vee B|\Gamma_i|\Gamma_1} \qquad\qquad \frac{\Gamma_0|\Gamma_i|\Gamma_1, A, B}{\Gamma_0|\Gamma_i|\Gamma_1, A \vee B}$$

$$\frac{\Gamma_0|\Gamma_i, A, B|\Gamma_1 \qquad \Gamma_0, A|\Gamma_i, A|\Gamma_1 \qquad \Gamma_0, B|\Gamma_i, B|\Gamma_1}{\Gamma_0|\Gamma_i, A \vee B|\Gamma_1}$$

$$\frac{\Gamma_0|\Gamma_i|\Gamma_1, A}{\Gamma_0, \neg A|\Gamma_i, B|\Gamma_1} \qquad \frac{\Gamma_0|\Gamma_i, A|\Gamma_1}{\Gamma_0|\Gamma_i, \neg A|\Gamma_1} \qquad \frac{\Gamma_0, A|\Gamma_i|\Gamma_1}{\Gamma_0|\Gamma_i|\Gamma_1, \neg A}$$

Logical consequence is defined as:

$$\Gamma \vdash_{K_3} \Delta \text{ iff the sequent } \Gamma|\Gamma|\Delta \text{ is derivable.}$$
$$\Gamma \vdash_{LP} \Delta \text{ iff the sequent } \Gamma|\Delta|\Delta \text{ is derivable.}$$

In this variety of pluralism, different consequence relations are set up by way of the sequents which are derivable in the calculus. Hjortland also suggest a 4-sided calculus to deal with FDE, K3 and LP, but does not develop it in detail. Since, according to Hjortland, the system he proposes does not rely on structural rules in order to define a consequence relation, it does not suffer from meaning variance of consequence relation. Hjortland does admit that this pluralism "works only for very limited cases" [Hjortland, 2013, p. 16].

One could argue that "even though the truth tables for K3 and LP are the same, that is a mere superficial similarity: the two logics still treat the middle value differently" [Hjortland, 2013, p. 17], and so the negation of K3 and of LP are different. Hjortland holds that "it seems equally cogent to maintain that we change only what is preserved in valid arguments, and not the antecedently given meanings of logical connectives" [Hjortland, 2013, p. 17]. A plurality of consequence relations exists, and a corresponding plurality of provability relations, but the proliferation happens within one and the same theory. There is thus pluralism without a plurality of logical theories.

Shapiro places Hjortland's pluralism in the same boat as Restall's. He notes that "none of the aforementioned proposals by Restall and Hjortland capture all of the logics of present interest" [Shapiro, 2014, p. 111], pointing out that maybe their underlying accounts of meaning are incorrect. He says it might be an option for "a pluralist to adopt one of these frameworks, and just rule out any logic that it does not capture. Those are not logics!" [Shapiro, 2014, p. 111], but for him, that is not the way for there is more to pluralism than this. Hjortland's pluralism is an instance of Cook's Substantial Logical Pluralism, but an uninteresting pluralism on Priest's account.

Summing up, the best move for a theory of logical pluralism seems to be to guarantee that the logical connectives share the same meaning in different logics, and what changes from logic to logic is the consequence relation. The most accepted way to do this is to have the connectives share the same rules and what changes from logic to logic is how the consequence relation is established. For Restall, what is needed are restrictions on number of conclusions/premisses, for Paoli structural rules such as Cut and Weakening and for Hjortland different derivable sequents. The differences and similarities of these approaches will be made more clear in Chapter Three, where they will be analyzed again in the light of B-entailment.

Having here introduced some views on logical pluralism of Shapiro, Beall and Restall, Restall alone, Paoli and Hjortland, in the next chapter B-entailment and cognitive attitude semantics are presented, non-determinism is discussed and a method for generating a sequent calculus for logics using B-entailment is introduced. Then, in Chapter Three, these two topics come together, B-entailment being analyzed in terms of the varieties of pluralisms presented in this chapter.

2 B-entailment Presented

This chapter presents B-entailment as a framework for expressing logical consequence relations which might simulate consequence relations present in the literature. The semantics uses cognitive attitudes rather than truth values as primitive. B-entailment is a four-place consequence relation which can deal with several kinds of reasoning about these cognitive attitudes. It is possible to simulate other logical consequence relations using B-entailment, which will be shown here for the consequence relations of p-entailment, q-entailment, t-entailment, and f-entailment. A method for creating sequent calculus rules for the logics that can be simulated using consequence relations based on B-entailment, as well as a discussion of non-determinism is presented.

2.1 Cognitive attitude semantics

Instead of interpreting sentences in terms of truth values, cognitive attitude semantics considers a society of agents who entertain cognitive attitudes concerning the acceptance and the rejection of a given piece of information. B-entailment is proposed as a framework for the notion of logical consequence relation that can simulate the reasoning behind the cognitive attitudes, COG, which are: Y (acceptance), ⅄ (non-acceptance), N (rejection) and И (non-rejection) [Blasio, 2016a]. Given a recursively formed language $\mathcal{S}$ formed by atomic formulas and any set of propositional connectives:

Definition 3. (Agent) An *agent* is a function s from sentences of a language to a set of cognitive attitudes.

Notation. $\mathsf{A}s{:}\varphi$ is read as: the agent s holds the attitude A towards the sentence φ.

Equivalently, it is possible to define the connectives over $\mathcal{S}$ in terms of usual truth tables using truth values. Consider the truth values $\mathbf{4} = \{\mathbf{f}, \bot, \top, \mathbf{t}\}$, in which these values are given in terms of the usual truth values 0 and 1, where $\mathbf{f} = \{0\}$, $\bot = \{\}$, $\top = \{0, 1\}$ and $\mathbf{t} = \{1\}$. Recall Definition 1 from Vocabulary (section 1.1 above), then a valuation v is a function from sentences of $\mathcal{S}$ to a truth value $x \in POW\{0, 1\}$. The general case of a valuation function is defined non-deterministically so that it maps to a set of truth values and not just one. A deterministic valuation is just one special case in which the sentences each map to a single truth value.

Definition 4. (Truth-function) A *truth function* is a function from a set of truth values to a subset of the set $\{0,1\}$, such that an n-ary connective f is interpreted as $\tilde{f} : \{1,0\}^n \to POW\{0,1\}$. Truth-functions are usually laid out in truth tables.

Given an agent s consulted about an atomic formula φ, the truth values of φ are attributed as such:

$\mathbf{f}$ if $\text{⅄}s{:}\varphi$ and $\mathsf{N}s{:}\varphi$, that is, s does not accept φ and rejects φ;

$\bot$ if $\text{⅄}s{:}\varphi$ and $\text{И}s{:}\varphi$, that is, s does not accept φ and does not reject φ;

$\top$ if $\mathsf{Y}s{:}\varphi$ and $\mathsf{N}s{:}\varphi$, that is, s accepts φ and rejects φ;

$\mathbf{t}$ if $\mathsf{Y}s{:}\varphi$ and $\text{И}s{:}\varphi$, that is, s accepts φ and does not reject φ.

Likewise it is possible to take the truth values of **4** as primitive and define the cognitive attitudes accordingly:

$$\mathsf{Y}s : \varphi \text{ iff } 1 \in v(\varphi) \text{ iff } v(\varphi) \in \{\top, \mathbf{t}\}$$

$$\text{⅄}s : \varphi \text{ iff } 1 \notin v(\varphi) \text{ iff } v(\varphi) \in \{\mathbf{f}, \bot\}$$

$$\mathsf{N}s : \varphi \text{ iff } 0 \in v(\varphi) \text{ iff } v(\varphi) \in \{\mathbf{f}, \top\}$$

$$\text{И}s : \varphi \text{ iff } 0 \notin v(\varphi) \text{ iff } v(\varphi) \in \{\bot, \mathbf{t}\}.$$

The choice to work with cognitive attitudes instead of truth-values is methodological, as it makes it simpler to work with B-entailment, since it is a four-place consequence relation and make use of one attitude in each place.

As a running example in this chapter, the semantics for the logic known as First-Degree Entailment (FDE) will be presented. Given $\varphi, \psi \in S$:

1. $\mathsf{Y}s{:}\neg\varphi$, if $\mathsf{N}s{:}\varphi$
2. $\mathsf{N}s{:}\neg\varphi$, if $\mathsf{Y}s{:}\varphi$
3. $\text{⅄}s{:}\neg\varphi$, if $\text{И}s{:}\varphi$
4. $\text{И}s{:}\neg\varphi$, if $\text{⅄}s{:}\varphi$

5. $\mathsf{Y}s{:}\varphi \wedge \psi$, if $\mathsf{Y}s{:}\varphi$ and $\mathsf{Y}s{:}\psi$
6. $\mathsf{N}s{:}\varphi \wedge \psi$, if $\mathsf{N}s{:}\varphi$ or $\mathsf{N}s{:}\psi$
7. $\text{⅄}s{:}\varphi \wedge \psi$, if $\text{⅄}s{:}\varphi$ or $\text{⅄}s{:}\psi$
8. $\text{И}s{:}\varphi \wedge \psi$, if $\text{И}s{:}\varphi$ and $\text{И}s{:}\psi$
9. $\mathsf{Y}s{:}\varphi \vee \psi$, if $\mathsf{Y}s{:}\varphi$ or $\mathsf{Y}s{:}\psi$
10. $\mathsf{N}s{:}\varphi \vee \psi$, if $\mathsf{N}s{:}\varphi$ and $\mathsf{N}s{:}\psi$
11. $\text{⅄}s{:}\varphi \vee \psi$, if $\text{⅄}s{:}\varphi$ and $\text{⅄}s{:}\psi$
12. $\text{И}s{:}\varphi \vee \psi$, if $\text{И}s{:}\varphi$ or $\text{И}s{:}\psi$

For FDE, the truth tables are equivalently defined as:

$\wedge$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$	$\bot$
$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$

$\vee$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
$\bot$	$\bot$	$\bot$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

	$\neg$
$\mathbf{f}$	$\mathbf{t}$
$\bot$	$\bot$
$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$

To see this, the following theorem is proven.

Theorem 1. *The truth-value semantics for First Degree Entailment is equivalent to is cognitive attitude semantics.*

Proof. Take Sem_1 as described by clauses 1 – 12 in terms of cognitive attitudes and Sem_2 as described by the above truth-tables in terms of **4**.

($\Rightarrow$) Given $s \in Sem_1$, it is possible to build some $v_s \in Sem_2$ such that, for every $\varphi \in \mathcal{S}$:

$$
\begin{aligned}
&(1)\ v_s(\varphi) = \mathbf{t} \text{ iff } s(\varphi) \in \{\mathsf{Y}, \text{И}\}\\
&(2)\ v_s(\varphi) = \top \text{ iff } s(\varphi) \in \{\mathsf{Y}, \mathsf{N}\}\\
&(3)\ v_s(\varphi) = \bot \text{ iff } s(\varphi) \in \{\text{⅄}, \text{И}\}\\
&(4)\ v_s(\varphi) = \mathbf{f} \text{ iff } s(\varphi) \in \{\text{⅄}, \mathsf{N}\}.
\end{aligned}
$$

To that effect, v_s is defined over atomic sentences according to (1)–(4) such that it also respects the truth tables above, and then one can check by induction on φ that (1)–(4) still hold for an arbitrary formula.

Case of $\neg$:

- Note that (A) $v_s(\neg\alpha) = \mathbf{t}$ iff [by checking the truth-tables above] (B) $v_s(\alpha) = \mathbf{f}$. Now, given (4) of the Induction Hypothesis, it is possible to see that (B) iff (C) ⅄$s{:}\alpha$ and N$s{:}\alpha$. Using clauses 2 and 3, one may see that (D) iff (E) И$s{:}\neg\alpha$ and Y$s{:}\neg\alpha$, which may be rewritten as (F) $s(\neg\alpha) = \{$Y, И$\}$.

- Note that (A) $v_s(\neg\alpha)) = \mathbf{f}$ iff [by checking the truth-tables above] (B) $v_s(\alpha) = \mathbf{t}$. Now, given (1) of the Induction Hypothesis, it is possible to see that (B) iff (C) Y$s{:}\alpha$ and И$s{:}\alpha$. Using clauses 2 and 3, one may see that (D) iff (E) N$s{:}\neg\alpha$ and ⅄$s{:}\neg\alpha$, which may be rewritten as (F) $s(\neg\alpha) = \{$N,⅄$\}$.

- Note that (A) $v_s(\neg\alpha) = \top$ iff [by checking the truth-tables above] (B) $v_s(\alpha) = \top$. Now, given (2) of the Induction Hypothesis, it is possible to see that (B) iff (C) Y$s{:}\alpha$ and N$s{:}\alpha$. Using clauses 1 and 2, one may see that (D) iff (E) N$s{:}\neg\alpha$ and Y$s{:}\neg\alpha$, which may be rewritten as (F) $s(\neg\alpha) = \{$Y, N$\}$.

- Note that (A) $v_s(\neg\alpha)) = \bot$ iff [by checking the truth-tables above] (B) $v_s(\alpha) = \bot$. Now, given (3) of the Induction Hypothesis, it is possible to see that (B) iff (C) ⅄$s{:}\alpha$ and И$s{:}\alpha$. Using clauses 3 and 4, one may see that (D) iff (E) И$s{:}\neg\alpha$ and ⅄$s{:}\neg\alpha$, which may be rewritten as (F) $s(\neg\alpha) = \{$⅄, И$\}$.

Case of $\wedge$:

- Note that (A) $v_s(\alpha \wedge \beta) = \mathbf{t}$ iff [by checking the truth-tables above] (B) $v_s(\alpha) = \mathbf{t} = v_s(\beta)$. Now, given (1) of the Induction Hypothesis, it is possible to see that (B) iff (C) $s(\alpha) = \{$Y, И$\} = s(\beta)$. One can conclude that (C) iff (D) Y$s{:}\alpha$ and Y$s{:}\beta$, and И$s{:}\alpha$ and И$s{:}\beta$. Using clauses 5 and 8, one may see that (D) iff (E) Y$s{:}\alpha \wedge \beta$ and И$s{:}\alpha \wedge \beta$, which may be rewritten as (F) $s(\alpha \wedge \beta) = \{$Y, И$\}$.

- Note that (A) $v_s(\alpha \wedge \beta) = \mathbf{f}$ iff [by checking the truth-tables above] (B): (i) $v_s(\alpha) = \mathbf{f}$, or (ii) $v_s(\beta) = \mathbf{f}$, or (iii) $v_s(\alpha) = \top$ and $v_s(\beta) = \bot$, or (iv) $v_s(\alpha) = \bot$ and $v_s(\beta) = \top$. Now, given (1) – (3) of the Induction Hypothesis, it is possible to see that (B) iff (C): (v) $s(\alpha) = \{$⅄, N$\}$, or (vi) $s(\beta) = \{$⅄, N$\}$, or (vii) $s(\alpha) = \{$Y, N$\}$ and $s(\beta) = \{$⅄, И$\}$, or (vii) $s(\alpha) = \{$⅄, И$\}$ and $s(\beta) = \{$Y, N$\}$. By sorting out the cases, one can conclude that (C) iff (D) N$s{:}\alpha$ or N$s{:}\beta$, and ⅄$s{:}\alpha$ or ⅄$s{:}\beta$. Using clauses 6 and 7, one may see that (D) iff (E) $\mathsf{N}s{:}\alpha \wedge \beta$ and $⅄s{:}\alpha \wedge \beta$, which may be rewritten as (F) $s(\alpha \wedge \beta) = \{⅄, \mathsf{N}\}$.

- Note that (A) $v_s(\alpha \wedge \beta) = \top$ iff [by checking the truth-tables above] (B): (i) $v_s(\alpha) = \top$ and $v_s(\beta) = \top$, or (ii) $v_s(\alpha) = \top$ and $v_s(\beta) = \mathbf{t}$, or (iii) $v_s(\alpha) = \mathbf{t}$ and $v_s(\beta) = \top$. Now, given (2) and (4) of the Induction Hypothesis, it is possible to see that (B) iff (C): (iv) $s(\alpha) = \{\mathsf{Y}, \mathsf{N}\}$ and $s(\beta) = \{\mathsf{Y}, \mathsf{N}\}$, or (v) $s(\alpha) = \{\mathsf{Y}, \mathsf{N}\}$ and $s(\beta) = \{\mathsf{Y}, И\}$, or (vi) $s(\alpha) = \{\mathsf{Y}, И\}$ and $s(\beta) = \{\mathsf{Y}, \mathsf{N}\}$. By sorting out the cases, one can conclude that (C) iff (D) $\mathsf{Y}s{:}\alpha$ and $\mathsf{Y}s{:}\beta$ and (either $\mathsf{N}s{:}\alpha$ or $\mathsf{N}s{:}\beta$). Using clauses 5 and 6, one may see that (D) iff (E) $\mathsf{Y}s{:}\alpha \wedge \beta$ and $\mathsf{N}s{:}\alpha \wedge \beta$, which may be rewritten as (F) $s(\alpha \wedge \beta) = \{\mathsf{Y}, \mathsf{N}\}$.

- Note that (A) $v_s(\alpha \wedge \beta) = \bot$ iff [by checking the truth-tables above] (B): (i) $v_s(\alpha) = \bot = v_s(\beta)$, or (ii) $v_s(\alpha) = \bot$ and $v_s(\beta) = \mathbf{t}$, or (iii) $v_s(\alpha) = \mathbf{t}$ and $v_s(\beta) = \bot$. Now, given (1) and (3) of the Induction Hypothesis, it is possible to see that (B) iff (C): (iv) $s(\alpha) = \{⅄, И\} = s(\beta)$, or (v) $s(\alpha) = \{⅄, И\}$ and $s(\beta) = \{\mathsf{Y}, И\}$, or (vi) $s(\alpha) = \{\mathsf{Y}, И\}$ and $s(\beta) = \{⅄, И\}$. By sorting out the cases, one can conclude that (C) iff (D) $Иs{:}\alpha$ and $Иs{:}\beta$ and (either $⅄s{:}\alpha$ or $⅄s{:}\beta$). Using clauses 7 and 8, one may see that (D) iff (E) $Иs{:}\alpha \wedge \beta$ and $⅄s{:}\alpha \wedge \beta$, which may be rewritten as (F) $s(\alpha \wedge \beta) = \{⅄, И\}$.

Case of $\vee$:

- Note that (A) $v_s(\alpha \vee \beta) = \mathbf{t}$ iff [by checking the truth-tables above] (B): (i) $v_s(\alpha) = \mathbf{t}$, or (ii) $v_s(\beta) = \mathbf{t}$, or (iii) $v_s(\alpha) = \top$ and $v_s(\alpha) = \bot$, or (iv) $v_s(\alpha) = \bot$ and $v_s(\alpha) = \top$. Now, given (1)–(4) of the Induction Hypothesis, it is possible to see that (B) iff (C): (v) $s(\alpha) = \{\mathsf{Y}, И\}$, or (vi) $s(\beta) = \{\mathsf{Y}, И\}$, or (vii) $s(\alpha) = \{\mathsf{Y}, \mathsf{N}\}$ and $s(\beta) = \{⅄, И\}$, or (viii) $s(\alpha) = \{⅄, И\}$ and $s(\beta) = \{\mathsf{Y}, \mathsf{N}\}$. By sorting out the cases, one can conclude that (C) iff (D) $\mathsf{Y}s{:}\alpha$ or $\mathsf{Y}s{:}\beta$, and $Иs{:}\alpha$ or $Иs{:}\beta$. Using clauses 9 and 12, one may see that (D) iff (E) $\mathsf{Y}s{:}\alpha \vee \beta$ and $Иs{:}\alpha \vee \beta$, which may be rewritten as (F) $s(\alpha \vee \beta) = \{\mathsf{Y}, И\}$.

- Note that (A) $v_s(\alpha \vee \beta) = \mathbf{f}$ iff [by checking the truth-tables above] (B): $v_s(\alpha) = \mathbf{f} = v_s(\beta)$. Now, given (4) of the Induction Hypothesis, it is possible to see that (B) iff (C) $s(\alpha) = \{⅄, \mathsf{N}\} = s(\beta)$. One can conclude that (C) iff (D) $⅄s{:}\alpha$ and $⅄s{:}\beta$, and

$\mathsf{N}s{:}\alpha$ and $\mathsf{N}s{:}\beta$. Using clauses 10 and 11, one may see that (D) iff (E) $\text{⅄}s{:}\alpha \vee \beta$ and $\mathsf{N}s{:}\alpha \vee \beta$, which may be rewritten as (F) $s(\alpha \vee \beta) = \{\text{⅄}, \mathsf{N}\}$.

- Note that (A) $v_s(\alpha \vee \beta) = \top$ iff [by checking the truth-tables above] (B): (i) $v_s(\alpha) = \top$ and $v_s(\beta) = \mathbf{f}$, or (ii) $v_s(\alpha) = \top$ and $v_s(\beta) = \top$, or (iii) $v_s(\alpha) = \mathbf{f}$ and $v_s(\beta) = \top$. Now, given (2) and (4) of the Induction Hypothesis, it is possible to see that (B) iff (C): (iv) $s(\alpha) = \{\mathsf{Y}, \mathsf{N}\}$ and $s(\beta) = \{\text{⅄}, \mathsf{N}\}$, or (v) $s(\alpha) = \{\mathsf{Y}, \mathsf{N}\}$ and $s(\beta) = \{\mathsf{Y}, \mathsf{N}\}$, or (vi) $s(\alpha) = \{\text{⅄}, \mathsf{N}\}$ and $s(\beta) = \{\mathsf{Y}, \mathsf{N}\}$. By sorting out the cases, one can conclude that (C) iff (D) $\mathsf{N}s{:}\alpha$ and $\mathsf{N}s{:}\beta$ and (either $\mathsf{Y}s{:}\alpha$ or $\mathsf{Y}s{:}\beta$). Using clauses 9 and 10, one may see that (D) iff (E) $\mathsf{Y}s{:}\alpha \vee \beta$ and $\mathsf{N}s{:}\alpha \vee \beta$, which may be rewritten as (F) $s(\alpha \vee \beta) = \{\mathsf{Y}, \mathsf{N}\}$.

- Note that (A) $v_s(\alpha \vee \beta) = \bot$ iff [by checking the truth-tables above] (B): (i) $v_s(\alpha) = \bot$ and $v_s(\beta) = \mathbf{f}$, or (ii) $v_s(\alpha) = \bot$ and $v_s(\beta) = \bot$, or (iii) $v_s(\alpha) = \mathbf{f}$ and $v_s(\beta) = \bot$. Now, given (3) and (4) of the Induction Hypothesis, it is possible to see that (B) iff (C): (iv) $s(\alpha) = \{\text{⅄}, \text{И}\}$ and $s(\beta) = \{\text{⅄}, \mathsf{N}\}$, or (v) $s(\alpha) = \{\text{⅄}, \text{И}\}$ and $s(\beta) = \{\text{⅄}, \text{И}\}$, or (vi) $s(\alpha) = \{\text{⅄}, \mathsf{N}\}$ and $s(\beta) = \{\text{⅄}, \text{И}\}$. By sorting out the cases, one can conclude that (C) iff (D) $\text{⅄}s{:}\alpha$ and $\text{⅄}s{:}\beta$ and (either $\text{И}s{:}\alpha$ or $\text{И}s{:}\beta$). Using clauses 11 and 12, one may see that (D) iff (E) $\text{⅄}s{:}\alpha \vee \beta$ and $\text{И}s{:}\alpha \vee \beta$, which may be rewritten as (F) $s(\alpha \vee \beta) = \{\text{⅄}, \text{И}\}$.

($\Leftarrow$) Given $v \in \mathit{Sem}_2$ it is possible to build $s_v \in \mathit{Sem}_1$ such that that for all $\varphi \in S$:

$$\mathsf{Y}s_v{:}\varphi \text{ iff } v(\varphi) \in \{\top, \mathbf{t}\}$$
$$\text{⅄}s_v{:}\varphi \text{ iff } v(\varphi) \in \{\bot, \mathbf{f}\}$$
$$\mathsf{N}s_v{:}\varphi \text{ iff } v(\varphi) \in \{\top, \mathbf{f}\}$$
$$\text{И}s_v{:}\varphi \text{ iff } v(\varphi) \in \{\bot, \mathbf{t}\}$$

Case of $\neg$:

- $\mathsf{Y}s_v{:}\neg\varphi$ iff $v(\neg\varphi) \in \{\top, \mathbf{t}\}$, by the definition of s_v. Then by the truth table of negation, $v(\varphi) \in \{\top, \mathbf{f}\}$. In which case $\mathsf{N}s_v{:}\varphi$, by the definition of s_v.

- $\mathsf{N}s_v{:}\neg\varphi$ iff $v(\neg\varphi) \in \{\top, \mathbf{f}\}$, by the definition of s_v. Then by the truth table of negation, $v(\varphi) \in \{\top, \mathbf{t}\}$. In which case $\mathsf{Y}s_v{:}\varphi$, by the definition of s_v.

- $\text{⅄}s_v{:}\neg\varphi$ iff $v(\neg\varphi) \in \{\bot, \mathbf{f}\}$, by the definition of s_v. Then by the truth table of negation, $v(\varphi) \in \{\bot, \mathbf{t}\}$. In which case $\text{И}s_v{:}\varphi$, by the definition of s_v.

- И$s_v{:}\neg\varphi$ iff $v(\neg\varphi) \in \{\bot, \mathbf{t}\}$, by the definition of s_v. Then by the truth table of negation, $v(\varphi) \in \{\bot, \mathbf{f}\}$. In which case ⅄$s_v{:}\varphi$, by the definition of s_v.

Case of $\wedge$:

- $\mathsf{Y}s_v{:}\varphi_1 \wedge \varphi_2$ iff $v(\varphi_1 \wedge \varphi_2) \in \{\top, \mathbf{t}\}$, by the definition of s_v. Then by the truth table of conjunction, $v(\varphi_1) \in \{\top, \mathbf{t}\}$ and $v(\varphi_2) \in \{\top, \mathbf{t}\}$. In which case, $\mathsf{Y}s_v{:}\varphi_1$ and $\mathsf{Y}s_v{:}\varphi_2$, by the definition of s_v.

- $\mathsf{N}s_v{:}\varphi_1 \wedge \varphi_2$ iff $v(\varphi_1 \wedge \varphi_2) \in \{\top, \mathbf{f}\}$, by the definition of s_v. Then by the truth table of conjunction, $v(\varphi_1) \in \{\top, \mathbf{f}\}$ and $v(\varphi_2) \in \{\top, \mathbf{f}\}$. In which case, $\mathsf{N}s_v{:}\varphi_1$ and $\mathsf{N}s_v{:}\varphi_2$, by the definition of s_v.

- ⅄$s_v{:}\varphi_1 \wedge \varphi_2$ iff $v(\varphi_1 \wedge \varphi_2) \in \{\bot, \mathbf{f}\}$, by the definition of s_v. Then by the truth table of conjunction, $v(\varphi_1) \in \{\bot, \mathbf{f}\}$ and $v(\varphi_2) \in \{\bot, \mathbf{f}\}$. In which case, ⅄$s_v{:}\varphi_1$ and ⅄$s_v{:}\varphi_2$, by the definition of s_v.

- И$s_v{:}\varphi_1 \wedge \varphi_2$ iff $v(\varphi_1 \wedge \varphi_2) \in \{\bot, \mathbf{t}\}$, by the definition of s_v. Then by the truth table of conjunction, $v(\varphi_1) \in \{\bot, \mathbf{t}\}$ and $v(\varphi_2) \in \{\bot, \mathbf{t}\}$. In which case, И$s_v{:}\varphi_1$ and И$s_v{:}\varphi_2$, by the definition of s_v.

Case of $\vee$:

- $\mathsf{Y}s_v{:}\varphi_1 \vee \varphi_2$ iff $v(\varphi_1 \vee \varphi_2) \in \{\top, \mathbf{t}\}$, by the definition of s_v. Then by the truth table of disjunction, $v(\varphi_1) \in \{\top, \mathbf{t}\}$ or $v(\varphi_2) \in \{\top, \mathbf{t}\}$. In which case, $\mathsf{Y}s_v{:}\varphi_1$ or $\mathsf{Y}s_v{:}\varphi_2$, by the definition of s_v.

- $\mathsf{N}s_v{:}\varphi_1 \vee \varphi_2$ iff $v(\varphi_1 \vee \varphi_2) \in \{\top, \mathbf{f}\}$, by the definition of s_v. Then by the truth table of disjunction, $v(\varphi_1) \in \{\top, \mathbf{f}\}$ or $v(\varphi_2) \in \{\top, \mathbf{f}\}$. In which case, $\mathsf{N}s_v{:}\varphi_1$ or $\mathsf{N}s_v{:}\varphi_2$, by the definition of s_v.

- ⅄$s_v{:}\varphi_1 \vee \varphi_2$ iff $v(\varphi_1 \vee \varphi_2) \in \{\bot, \mathbf{f}\}$, by the definition of s_v. Then by the truth table of disjunction, $v(\varphi_1) \in \{\bot, \mathbf{f}\}$ or $v(\varphi_2) \in \{\bot, \mathbf{f}\}$. In which case, ⅄$s_v{:}\varphi_1$ or ⅄$s_v{:}\varphi_2$, by the definition of s_v.

- И$s_v{:}\varphi_1 \vee \varphi_2$ iff $v(\varphi_1 \vee \varphi_2) \in \{\bot, \mathbf{t}\}$, by the definition of s_v. Then by the truth table of disjunction, $v(\varphi_1) \in \{\bot, \mathbf{t}\}$ or $v(\varphi_2) \in \{\bot, \mathbf{t}\}$. In which case, И$s_v{:}\varphi_1$ or И$s_v{:}\varphi_2$, by the definition of s_v.

□

2.2 Semantical matrices

In what follows, semantical matrices will be used to characterize consequence relations. In this section semantical matrices are introduced, first by way of a standard matrix, and then a generalization is presented. Based on this, B-entailment is finally defined.

A standard semantical matrix is a structure $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a set of truth values, $\mathcal{D}$ is the set of designated truth values ($\mathcal{D} \subseteq \mathcal{V}$), and $\mathcal{O}$ is the set of truth functions of the connectives of a language $\mathcal{S}$. For now, $\mathcal{O}$ is deterministic, which means that for each input, each truth-function maps to only one truth value, instead of a set of them. The logical consequence relation based on $\mathfrak{M}$ is such that:

Definition 5. (Tarskian -consequence) $\Gamma \models \varphi$ iff there is no v such that $v(\Gamma) \subseteq \mathcal{D}$ and $v(\varphi) \in \mathcal{V} - \mathcal{D}$.

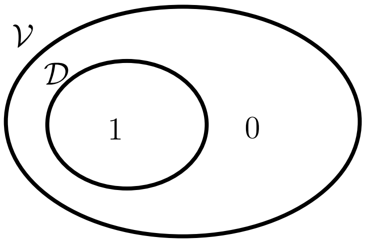

Figure 1: Standard semantical matrix with $\mathcal{V} = \{0, 1\}$.

In this simple case, there there is only one distinguished set, and so if there were more truth values in $\mathcal{D}$, this matrix could not tell them apart. It is possible to define a logical relation of backwards preservation of falsity, that is, a preservation from the conclusion to the premisses. In the case where there is only one distinguished set of values, however, the backward preservation of falsity is the same as the preservation of truth from premisses to the conclusion, because the definition of preservation of falsity would be the same as the preservation of not-truth.

Generalized semantical matrices (also called symmetrical matrices) [Shramko and Wansing, 2011] will offer a way of dealing with truth values when one wishes that the association of the concepts of "not true" and "false" come apart. Instead of talking only about the preservation of the truth from the premisses to the conclusion, it is also possible to talk about the preservation of falsity from the conclusion to the premisses in a way that these two consequence relations do not collapse into one another, and similarly for the preservation of non-falsity. A generalized semantical matrix is of the form $\mathfrak{G} = \langle \mathcal{V}, \mathcal{D}, \mathcal{U}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a set of truth values, $\mathcal{D}$ is the set of designated values, $\mathcal{U}$ is the set of undesignated (or antidesignated) values, and $\mathcal{O}$ is the set of truth functions of the connectives of a language $\mathcal{S}$. Based on $\mathfrak{G}$, different logical consequence relations can be defined, such as:

Definition 6. (+-consequence) $\Gamma \models^{+} \varphi$ iff there is no v such that $v(\Gamma) \subseteq \mathcal{D}$ and $v(\varphi) \in \mathcal{U}$.

Definition 7. (−-consequence) $\Gamma \models^{-} \varphi$ iff there is no v such that $v(\Gamma) \subseteq \mathcal{V} - \mathcal{D}$ and $v(\varphi) \in \mathcal{V} - \mathcal{U}$.

By varying which values are taken as designated and un-designated, as well as by defining consequence relation as preservation of either set or a combination of both, it is possible to create different consequence relations.

B-entailment is defined by taking as primitive not the truth values in the usual sense, but instead the cognitive attitudes. Considering the acceptance and rejection of information by agents, a generalized matrix $\mathscr{B} = \{COG, \mathcal{D}^{\mathcal{B}}, \mathcal{U}^{\mathcal{B}}, \mathcal{O}\}$ may be defined such that, $\mathcal{D}^{\mathcal{B}} = \mathsf{Y} = \{\mathbf{t}, \top\}$ and $\mathcal{U}^{\mathcal{B}} = \mathsf{N} = \{\mathbf{f}, \bot\}$ are two sets of distinguished values, and as before $\mathcal{O}$ is the set of operations interpreting the language $\mathcal{S}$. Relative to COG, we define ⅄ as the complement of Y and И as the complement of N.

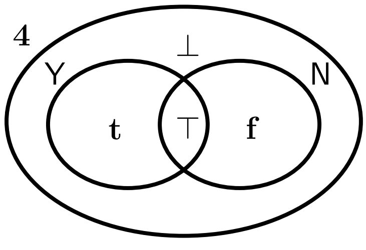

Figure 2: Generalized matrix for *B*-entailment.

B-entailment is the entailment relation defined as:

Definition 8. (*B*-entailment) $\frac{\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi}$ if there is no agent s such that Y$s{:}\Gamma$ and ⅄$s{:}\Delta$ and N$s{:}\Phi$ and И$s{:}\Psi$.

In other words, there is no agent s such that s accepts all sentences of Γ and s does not accept any sentence of Δ and s rejects all sentences of Φ and s does not reject any sentence of Ψ.

Using the framework of B-entailment presented, it is possible to simulate other logical consequence relations, which will be done in the next section.

2.3 Refinements

This section will explain how to refine B-entailment in order to simulate other consequence relations, in particular, p-entailment, q-entailment, t-entailment and f-entailment. In this sense, a "refinement" of a consequence relation is a consequence relation that is more restrictive than the previous one but that is still based on the same semantic matrix. By setting restrictions on B-entailment, these consequence relations are shown to be each a particular case of B-entailment. These refinements of B-entailment are simulations of other entailment relations in the sense that they are equivalent. These terms are defined bellow.

Definition 9. (Refinement) Given two semantical matrices $\mathfrak{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathfrak{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ for a language $\mathcal{S}$, $\mathfrak{M}_1$ is called a *refinement* of $\mathfrak{M}_2$ if $\mathcal{V}_2 \subseteq \mathcal{V}_1$, $\mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{V}_2$ and $\tilde{f}_{\mathfrak{M}_2}(\vec{x}) \subseteq \tilde{f}_{\mathfrak{M}_1}(\vec{x})$, for every n-ary connective f of $\mathcal{S}$ and every $x \in \mathcal{V}_2^n$.

Definition 10. (Simulation) Given two consequence relations $\vdash_1$ and $\vdash_2$, it is said that $\vdash_2$ is a *simulation* of $\vdash_1$ if it validates the same arguments than $\vdash_1$.

The logical consequence relations of t-entailment and f-entailment are tarskian relations [Shramko and Wansing, 2011]. The first one preserves the truth from the premisses to the conclusion (like $\models^+$ from the previous section), while the second one preserves the falsity from the conclusion to the premisses (much like $\models^-$ from the previous section). The semantical matrix for t-entailment is such that $\mathfrak{M}^t = \langle \mathbf{4}, \mathsf{Y}, \mathcal{O} \rangle$ while the one for f-entailment is such that $\mathfrak{M}^f = \langle \mathbf{4}, \mathsf{N}, \mathcal{O} \rangle$.

The consequence relations t-entailment and f-entailment can be defined using cognitive attitudes and expressed using B-entailment based on $\mathscr{B}$ in the following way[1]:

Definition 11. (t-consequence) $\Gamma \models^t \Phi$ iff there is no s, such that $\mathsf{Y}s{:}\Gamma$ and $\curlywedge s{:}\Phi$ iff $\frac{\cdot}{\Gamma}\Big|\frac{\Phi}{\cdot}$.

1 The notation $\cdot$ indicates that the set of formulas is empty.

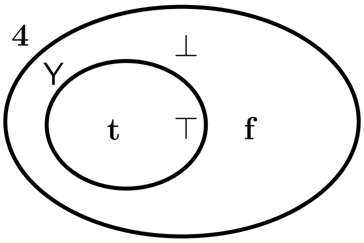

Figure 3: Matrix for t-entailment.

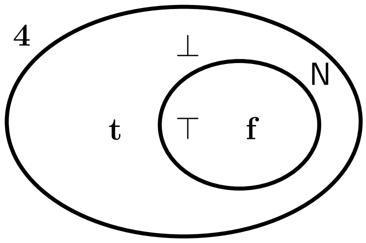

Figure 4: Matrix for f-entailment.

Definition 12. (f-consequence) $\Gamma \vDash^f \Phi$ iff there is no s, such that $\not\mathsf{N}s{:}\Gamma$ and $\mathsf{N}s{:}\Phi$ iff $\frac{\Gamma}{\cdot}\Big|\frac{\cdot}{\Phi}$.

By t-consequence, and recalling that Y is the complement of ⅄, since there is no agent s that both accepts Γ and does not accept φ, that means that if an agent accepts the premisses of the argument, she must also accept the conclusion. Acceptance (and so the truth-value $\mathbf{t}$) is thus preserved from the premisses to the conclusion, as desired. By f-consequence, since there is no agent s that both does not reject Γ and rejects φ, that means that if an agent rejects the conclusion, she must also reject (and thus preserve the truth-value $\mathbf{f}$) some of the premisses. Thus rejection is preserved from the conclusion to the premisses. If both these notions of consequence are taken together, the resulting entailment relation is that of classical logic.

Theorem 2. *$\models^t$ and $\models^f$ together recover the tarskian consequence (Definition 4).*

Proof. Suppose $\Gamma \models^t \varphi$ and $\Gamma \models^f \varphi$. By t-consequence, there is no s, such that $\mathsf{Y}s{:}\Gamma$ and ⅄$s{:}\varphi$. By f-consequence, there is no s, such that $\not\mathsf{N}s{:}\Gamma$ and $\mathsf{N}s{:}\varphi$. Taking these together, there is no v such that $v(\Gamma) = \{\top, \mathbf{t}\} \cap \{\bot, \mathbf{t}\} = \mathbf{t}$ and $v(\varphi) = \{\bot, \mathbf{f}\} \cap \{\mathbf{f}, \top\} = \mathbf{f}$. Therefore, there is no v such that $v(\Gamma)$ is designated and $v(\varphi)$ is not designated, which is what the Tarskian entailment demands. □

Two other consequence relations that can be simulated using B-entailment are p-entailment [Frankowski, 2004] and q-entailment [Malinowski, 1990]. In a usual interpretation, the p stands for plausible, as it intends to relay a relation of plausibility between the premisses and the conclusion: if one accepts the premisses, then the conclusion should not be rejected. One does not go as far as to accept the conclusion, but considers it plausible enough not to be rejected. Regarding q-entailment, one reading for q is that it stands for quasi, as it is a relation in which if one does not reject the premisses, the conclusion should be accepted. This is deemed not a full entailment, for instead of accepting the premisses (which would make it a full entailment), one is content in not rejecting them. This consequence relation offers a way of reasoning which can be viewed as similar to how science works. As long as a hypothesis has not been ruled out (and hence not rejected), inferences based on it are accepted.

These entailment relations can be associated to the matrix $\mathscr{B}$ [Blasio, 2016a].

Definition 13. (p-entailment) $\Gamma \models^p \varphi$ iff there is no agent s such that $\mathsf{Y}s{:}\Gamma$ and $\mathsf{N}s{:}\varphi$.

Definition 14. (q-entailment) $\Gamma \models^q \varphi$ iff there is no agent s such that $\text{И}s{:}\Gamma$ and $\text{⅄}s{:}\varphi$.

As before, p-entailment and q-entailment can be defined using cognitive attitudes as forms of the B-entailment in the following way:

Definition 15. (p-consequence) $\Gamma \vDash^p \Phi$ iff there is no s, such that $\mathsf{Y}s{:}\Gamma$ and $\mathsf{N}s{:}\Phi$ iff $\frac{\cdot}{\Gamma}\Big|\frac{\cdot}{\Phi}$.

Definition 16. (p-restraint) There is no s such that $\text{⅄}s{:}\alpha$ and $\text{И}s{:}\alpha$ iff $\frac{\alpha}{\cdot}\Big|\frac{\alpha}{\cdot}$.

Definition 17. (q-consequence) $\Gamma \vDash^q \Phi$ iff there is no s, such that $\text{И}s{:}\Gamma$ and $\text{⅄}s{:}\Phi$ iff $\frac{\Gamma}{\cdot}\Big|\frac{\Phi}{\cdot}$.

Definition 18. (q-restraint) There is no s such that $\mathsf{Y}s{:}\alpha$ and $\mathsf{N}s{:}\alpha$ iff $\frac{\cdot}{\alpha}\Big|\frac{\cdot}{\alpha}$.

By the clause of p-restraint, since there is no agent s that both does not accept α and does not reject α, this means that it cannot be the case that $s(\alpha) \in \{\mathbf{f}, \bot\}$ and $s(\alpha) \in \{\bot, \mathbf{t}\}$. Thus, $s(\alpha) \neq \bot$, restricting the values from $\mathbf{4}$ that a sentence α can take by eliminating $\bot$[2]. The p-consequence clause states the form which the argument takes: it cannot be the case that if the premisses are accepted then the conclusion is rejected, so it must be not rejected.

By the clause of q-restraint, since there is no agent s that both accepts α and rejects α, this means that it cannot be the case that $s(\alpha) \in \{\top, \mathbf{t}\}$ and $s(\alpha) \in \{\mathbf{f}, \top\}$. Thus, $s(\alpha) \neq \top$, which restricts the values from $\mathbf{4}$ that a sentence α can take by eliminating $\top$[3]. The q-consequence clause states the form which the argument takes: if it cannot be the case that if the premisses are non-rejected then the conclusion is non-accepted, so the conclusion must the accepted.

[2] It's due notice that this restriction with the presence of a negation operator, excludes paracompleteness.

[3] It's due notice that this restriction with the presence of a negation operator, excludes paraconsistency.

As an example of using these entailment-relations, the logic FDE makes use of the t-entailment [Arieli and Avron, 1998] that can be simulated using B-entailment [Blasio, 2016a]. This logic has been introduced by J. Michael Dunn [Dunn, 1976] and Nuel D. Belnap [Belnap, 1977]. FDE finds application in computer science, motivated by a desire to provide for a reasoning mechanism that could turn a computer into a question-answering system on the basis of deduction instead of just reciting what is in its memory bank. What is more, the system would be able to obtain data from multiple reliable sources which might be inconsistent with each other. In this way, the computer could keep on working even when finding such inconsistency.

In this section, $\vdots\!\mid\!\vdots$ was refined in order to simulate other logical consequence relations. To do so, restraints were added to the effect that certain truth-values were eliminated as well as different notions consequence relation were presented. Some logical consequence relations present in the literature were simulated. In the subsequent section, a different aspect of B-entailment's semantics is explored, namely, its non-determinism.

2.4 Non-determinism

As stated in the section Cognitive attitude semantics (section 2.1 above), the general framework of B-entailment has a non-deterministic semantics, which so far in this thesis has been explored only in the specific case in which the truth functions are deterministic. In this section, the non-deterministic aspect of B-entailment is explored, so that in the next section a method for generating a sequent calculus can be presented, in which non-deterministic functions give rise to the sequent rules. Here it will be shown how to determine a truth-table with two truth values, $\mathbf{t}$ and $\mathbf{f}$, in a uni-dimensional multiple-conclusion relation. In the next section, a more general framework is presented which this is done for the two-dimensional relation of B-entailment with the truth-values of $\mathbf{4}$.

Recall the Definition 3 of truth-function from Cognitive attitude semantics (section 2.1 above). The definition allows for a connective to have a non-deterministic interpretation, which means that it can map each input to more than one truth-value. In the previous section, only the case in which truth functions map to a single truth-value was explored. More will be said here about the case in which the functions may not map to a singleton set of truth values. This feature of the semantics is interesting because it provides B-entailment with more expressive powers than the usual deterministic semantics. This is an interesting feature to analyze in the context of logical pluralism, which is done in Chapter Three bellow.

This kind of semantics might seem strange at first, but this is not necessarily so. To illustrate this, the first step is to start from a fully non-deterministic truth-table and then, given some rules, restrict the values of the output of the function. Given enough rules, the truth-table can become fully determined.

Fully non-deterministic truth-table:

β

	$\Rightarrow$	$\mathbf{f}$	$\mathbf{t}$
α	$\mathbf{f}$	$\mathbf{f}, \mathbf{t}$	$\mathbf{f}, \mathbf{t}$
	$\mathbf{t}$	$\mathbf{f}, \mathbf{t}$	$\mathbf{f}, \mathbf{t}$

Take the rules $\alpha, \alpha \Rightarrow \beta \vdash \beta$ and $\beta \vdash \alpha \Rightarrow \beta$, for example. The formulas on the left of $\vdash$ are read as true formulas and the ones on the right of $\vdash$ are read as false formulas. So the first rule says that it cannot be the case that $v(\alpha) = \mathbf{t}$ and $v(\beta) = \mathbf{f}$ and $v(\alpha \Rightarrow \beta) = \mathbf{t}$, while the second one says that it cannot be the case that $v(\beta) = \mathbf{t}$ and $v(\alpha \Rightarrow \beta) = \mathbf{f}$. These restrictions can be applied to the non-deterministic table and restrict the output values the following way:

Restriction from $\alpha, \alpha \Rightarrow \beta \vdash \beta$:

β

	$\Rightarrow$	$\mathbf{f}$	$\mathbf{t}$
α	$\mathbf{f}$	$\mathbf{f}, \mathbf{t}$	$\mathbf{f}, \mathbf{t}$
	$\mathbf{t}$	$\mathbf{f}, \cancel{\mathbf{t}}$	$\mathbf{f}, \mathbf{t}$

Restrictions from $\beta \vdash \alpha \Rightarrow \beta$:

β

	$\Rightarrow$	$\mathbf{f}$	$\mathbf{t}$
α	$\mathbf{f}$	$\mathbf{f}, \mathbf{t}$	$\cancel{\mathbf{f}}, \mathbf{t}$
	$\mathbf{t}$	$\mathbf{f}, \mathbf{t}$	$\cancel{\mathbf{f}}, \mathbf{t}$

Given these two rules, it is possible to obtain the intersection of the restrictions they impose on the truth tables above, and arrive at only one truth-table.

Definition 19. (Intersection of truth-tables) The *intersection of truth-tables* is a truth-table in each for each tuples which are the input of the function, take the intersection of the set of outputs, for each cell in the table.

The intersection of the previous truth tables is:

	$\Rightarrow$	β: $\mathbf{f}$	$\mathbf{t}$
α	$\mathbf{f}$	$\mathbf{f}, \mathbf{t}$	$\mathbf{t}$
	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$

The case in which $v(\alpha) = \mathbf{f}$ and $v(\beta) = \mathbf{f}$ remains non-deterministic, since the truth table was only partially determined by the given rules. In this shared fragment of classical and intuitionist implication, implication behaves the same way. So the difference between the implication in these two logics is not related to the cells of the truth-table that have been so far determined, but on the cell which was left un-determined.

It is possible to further determine the table with the rule $\vdash \alpha \Rightarrow \beta, \alpha$, which is read as not being the case that $v(\alpha \Rightarrow \beta) = \mathbf{f}$ and $v(\alpha) = \mathbf{f}$. The corresponding restriction on the non-deterministic truth-table is thus:

Restrictions from $\vdash \alpha \Rightarrow \beta, \alpha$:

	$\Rightarrow$	β: $\mathbf{f}$	$\mathbf{t}$
α	$\mathbf{f}$	$\cancel{\mathbf{f}}, \mathbf{t}$	$\cancel{\mathbf{f}}, \mathbf{t}$
	$\mathbf{t}$	$\mathbf{f}, \mathbf{t}$	$\mathbf{f}, \mathbf{t}$

Taking then the intersection of the restrictions imposed by all three determinants on a truth table, the new intersection becomes the truth-table for classical implication:

	$\Rightarrow$	β: $\mathbf{f}$	$\mathbf{t}$
α	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$
	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$

In this case, the implication becomes the classical implication; it cannot be the intuitionist implication because this last restriction on the truth table is not one which holds for intuitionist logic, since the added rule would be multiple-conclusion, which is not permitted in intuitionistic logic.

In the next section, a method for creating sequent rules will be presented using cognitive attitudes and in the broader framework of B-entailment. Rules can be seen as helping determine a non-deterministic truth table in specific cells.

2.5 Sequent calculus

This section showcases a method for presenting sequent rules intended for specifying the desired connectives of a logic, deterministically or otherwise. Then the sequent calculus for the logic FDE= $\langle \vdots \mid \vdots, \mathcal{S}\rangle$ based on B-entailment and language S is shown and also additional structural rules are presented which enable it to express the entailment relations discussed on the previous sections.

Sequent calculus is a system of formal proofs in logic, in which one starts from an initial sequent (which is a reflexive statement) and by use of the sequent rules, which aim at preserving truth through the steps of the derivation, reach a conclusion. Usually this procedure is effected from the bottom up, so that the reasoning proceeds from more complex sequents to simpler ones, until the initial sequent is reached. When dealing with forms of B-entailment, truth tables will be turned into rules dealing with the cognitive attitudes.

Following *The Connectives* [Humberstone, 2011], the method used to reach the sequent rules is based on determinants for a truth table. The method described in *The Connectives* is presented for two truth values ($\mathbf{t}$, $\mathbf{f}$) and for a one dimensional consequence relation ($\vdash$) and will be adapted here for the cognitive attitudes (Y,⅄, N, И) and for a two-dimensional consequence relation ($\vdots \mid \vdots$).

Definition 20. (Determinant) A *determinant* for a k-ary connective # will be a $(k+1)$-tuple of cognitive attitudes $\langle x_1, ..., x_k, ʎ\rangle$ which dictates that if an agent assigns to $B_1, ..., B_k$ the cognitive attitudes $x_1, ..., x_k$, respectively, then the formula $\#(B_1, ..., B_k)$ is assigned the cognitive attitude y, where ʎ is the complement of that cognitive attitude (where $x_i, y,$ ʎ $\in$ {Y,⅄, N, И}).

A determinant $\langle x_1, ..., x_k, ʎ\rangle$ is read as not being the case that an agent holds the cognitive attitudes $x_1, ..., x_k$ towards $B_1, ..., B_k$ and attitude ʎ towards $\#(B_1, ..., B_k)$.

Looking at a truth table for a connective, it is possible to write the determinants for this connective. Thus by systematically checking all possible combinations of cognitive attitudes as the input of the truth-table for $\wedge$ for FDE (from Cognitive attitude semantics) and the output produced by it, the determinants reached are: ⟨Y, Y, ⅄⟩, ⟨Y, ⅄, Y⟩, ⟨⅄, Y, Y⟩, ⟨⅄, ⅄, Y⟩, ⟨N, N, И⟩, ⟨N, И, И⟩, ⟨И, N, И⟩, ⟨И, И, N⟩. Again, by checking the highlighted lines on the truth table below it's possible to see how the determinant ⟨Y, Y, ⅄⟩ was reached. The first two items of determinants are Y and so one must check on the truth table below the output for α and β when their truth values are accepted, that is, check whether $v(\alpha \wedge \beta) \in \{\mathbf{t}, \top\}$. By checking, it is possible to see that the last item of the determinant will be ⅄, since by the truth-table the outputs are accepted (Y), and by the definition of determinant, the last item is the complement of the cognitive attitude in the table. The reading for this determinant is that it cannot be the case that the premisses are accepted and the conclusion not accepted.

	β				
	$\wedge$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
α	$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$	$\bot$
	$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$	$\top$
	$\mathbf{t}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$

The opposite may also be achieved. Start, say, from some determinants and a completely non-deterministic truth table based on $\mathbf{4}$ for a binary connective $\#$; and, by application of the each determinant, the output of the table becomes more restricted.

	ψ				
	$\#$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$
φ	$\bot$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$
	$\top$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$
	$\mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$	$\mathbf{f}, \bot, \top, \mathbf{t}$

Take then the following determinants: ⟨Y, Y, ⅄⟩, ⟨Y, ⅄, ⅄⟩, ⟨⅄, Y, ⅄⟩, ⟨⅄, ⅄, Y⟩, ⟨N, N, И⟩, ⟨N, И, N⟩, ⟨И, N, N⟩, ⟨И, И, N⟩. Each determinant restricts the truth-values for some cells in the truth-table in the following way.

By ⟨Y, Y, ⅄⟩:

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
φ	$\bot$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\top$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$
	$\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$

By ⟨Y, ⅄, ⅄⟩:

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$t\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
φ	$\bot$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\top$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$

By ⟨⅄, Y, ⅄⟩:

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$
φ	$\bot$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$	$\cancel{\mathbf{f}},\cancel{\bot},\top,\mathbf{t}$
	$\top$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$

By ⟨⅄, ⅄, Y⟩:

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f},\bot,\cancel{\top},\cancel{\mathbf{t}}$	$\mathbf{f},\bot,\cancel{\top},\cancel{\mathbf{t}}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
φ	$\bot$	$\mathbf{f},\bot,\cancel{\top},\cancel{\mathbf{t}}$	$\mathbf{f},\bot,\cancel{\top},\cancel{\mathbf{t}}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\top$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$

By ⟨N, N, И⟩:

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f},\cancel{\bot},\top,\cancel{\mathbf{t}}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\cancel{\bot},\top,\cancel{\mathbf{t}}$	$f,\bot,\top,\mathbf{t}$
φ	$\bot$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\top$	$\mathbf{f},\cancel{\bot},\top,\cancel{\mathbf{t}}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\cancel{\bot},\top,\cancel{\mathbf{t}}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$

By ⟨N, И, N⟩:

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$
φ	$\bot$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\top$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$
	$\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$

By ⟨И, N, N⟩:

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
φ	$\bot$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\top$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$

By ⟨И, И, N⟩:

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
φ	$\bot$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$
	$\top$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$
	$\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\mathbf{f},\bot,\top,\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$	$\cancel{\mathbf{f}},\bot,\cancel{\top},\mathbf{t}$

To arrive at only one truth table by way of all these determinants, take the intersection (as defined in Non-determinism):

ψ

	#	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
	$\mathbf{f}$	$\mathbf{f}$	$\bot$	$\top$	$\mathbf{t}$
φ	$\bot$	$\bot$	$\bot$	$\mathbf{t}$	$\mathbf{t}$
	$\top$	$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$
	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

It is then possible to see that these determinants describe the truth table of $\vee$ for FDE.

After establishing the determinants for a connective, a condition on $\vdots\!\mid\!\vdots$ can be induced by a determinant $\langle x_1, ..., x_k, \mathfrak{n}\rangle$ for a connective $\#$, based on which sequent rules can be elaborated. Writing the conditions is done in the following way:

- If x_i is Y put B_i in $\left.\frac{\cdot}{B_i}\right| :$
- If x_i is N, put B_i in $: \left| \frac{\cdot}{B_i}\right.$
- If x_i is ⅄, put B_i in $: \left| \frac{B_i}{\cdot}\right.$
- If x_i is И, put B_i in $\left.\frac{B_i}{\cdot}\right| :$
- If $\mathscr{h}$ is ⅄, put $\#(B_1, ..., B_k)$ in $: \left| \frac{\#(B_1,...,B_k)}{\cdot}\right.$
- If $\mathscr{h}$ is И, put $\#(B_1, ..., B_k)$ in $\left.\frac{\#(B_1,...,B_k)}{\cdot}\right| :$
- If $\mathscr{h}$ is Y, put $\#(B_1, ..., B_k)$ in $\left.\frac{\cdot}{\#(B_1,...,B_k)}\right| :$
- If $\mathscr{h}$ is N, put $\#(B_1, ..., B_k)$ in $: \left| \frac{\cdot}{\#(B_1,...,B_k)}\right.$

As an example of conditions for a connective, take the determinants for the conjunction of FDE, then the conditions for $\wedge$ are read as:

1. $\left.\frac{\cdot}{\alpha,\beta}\right|\frac{\alpha\wedge\beta}{\cdot}$ – it is not the case that $\mathsf{Y}s{:}\alpha$ and $\mathsf{Y}s{:}\beta$ and ⅄$s{:}\alpha \wedge \beta$.
2. $\left.\frac{\cdot}{\alpha,\alpha\wedge\beta}\right|\frac{\beta}{\cdot}$ – it is not the case that $\mathsf{Y}s{:}\alpha$ and ⅄$s{:}\beta$ and $\mathsf{Y}s{:}\alpha \wedge \beta$.
3. $\left.\frac{\cdot}{\beta,\alpha\wedge\beta}\right|\frac{\alpha}{\cdot}$ – it is not the case that ⅄$s{:}\alpha$ and $\mathsf{Y}s{:}\beta$ and $\mathsf{Y}s{:}\alpha \wedge \beta$.
4. $\left.\frac{\cdot}{\alpha\wedge\beta}\right|\frac{\alpha,\beta}{\cdot}$ – it is not the case that ⅄$s{:}\alpha$ and ⅄$s{:}\beta$ and $\mathsf{Y}s{:}\alpha \wedge \beta$.
5. $\left.\frac{\alpha\wedge\beta}{\cdot}\right|\frac{\cdot}{\alpha,\beta}$ – it is not the case that $\mathsf{N}s{:}\alpha$ and $\mathsf{N}s{:}\beta$ and И$s{:}\alpha \wedge \beta$.
6. $\left.\frac{\beta,\alpha\wedge\beta}{\cdot}\right|\frac{\cdot}{\alpha}$ – it is not the case that $\mathsf{N}s{:}\alpha$ and И$s{:}\beta$ and И$s{:}\alpha \wedge \beta$.
7. $\left.\frac{\beta,\alpha\wedge\beta}{\cdot}\right|\frac{\cdot}{\alpha}$ – it is not the case that И$s{:}\alpha$ and $\mathsf{N}s{:}\beta$ and И$s{:}\alpha \wedge \beta$.
8. $\left.\frac{\alpha,\beta}{\cdot}\right|\frac{\cdot}{\alpha\wedge\beta}$ – it is not the case that И$s{:}\alpha$ and И$s{:}\beta$ and $\mathsf{N}s{:}\alpha \wedge \beta$.

Taking these conditions for the determinants, it is possible to turn them into sequent rules in a system of sequent calculus. As an example, this will be done for FDE. First, the structural rules will be presented according to the sequent system for FDE using B-entailment as developed by Carolina Blasio [Blasio, 2016a] followed by derivations which turn these conditions on the logical rules of the system introduced here. Finally, it will be shown that the rules presented here are inter-derivable with the logical rules presented by Blasio.

B-sequents are expressions of the form $\frac{\Psi}{\Gamma}\big|\frac{\Delta}{\Phi}$ in which $\frac{\cdot}{\cdot}\big|\frac{\cdot}{\cdot}$ is a sequent symbol with four positions that will be called B-sequent and $\Gamma, \Psi, \Delta, \Phi \subseteq \mathcal{S}$ are finite sets of formulas from the language. Let $\bigwedge Cs{:}\Lambda$ be an abreviation for "the agent s presents the cognitive attitude C to λ_1 or ... or λ_n", where $\Lambda = \{\lambda_1, ..., \lambda_n\}$ and $n \in \mathbb{N}$. The B-sequent $\frac{\Psi}{\Gamma}\big|\frac{\Delta}{\Phi}$ is read as: there is no agent s such that $\bigwedge \mathsf{Y}s{:}\Gamma$ and $\bigwedge \text{⅄}s{:}\Delta$ and $\bigwedge \mathsf{N}s{:}\Phi$ and $\bigwedge \text{И}s{:}\Psi$.

For all atomic formulas $\alpha \in S$ and whatever finite sets of formulas Γ, Δ, Φ, Ψ, Γ', Δ', Φ', $\Psi' \subseteq S$, the system of B-sequents for FDE has the following structural rules:

Reflexivities

$$\dfrac{}{\frac{\cdot}{\alpha}\big|\frac{\alpha}{\cdot}}\ ref_t \qquad \dfrac{}{\frac{\alpha}{\cdot}\big|\frac{\cdot}{\alpha}}\ ref_f$$

Weakening

$$\dfrac{\frac{\Psi}{\Gamma}\big|\frac{\Delta}{\Phi}}{\frac{\Psi',\Psi}{\Gamma',\Gamma}\big|\frac{\Delta,\Delta'}{\Phi,\Phi'}}\ weak$$

Cuts

$$\dfrac{\frac{\Psi}{\alpha,\Gamma}\big|\frac{\Delta}{\Phi} \quad \frac{\Psi}{\Gamma}\big|\frac{\Delta,\alpha}{\Phi}}{\frac{\Psi}{\Gamma}\big|\frac{\Delta}{\Phi}}\ cut_t \qquad \dfrac{\frac{\alpha,\Psi}{\Gamma}\big|\frac{\Delta}{\Phi} \quad \frac{\Psi}{\Gamma}\big|\frac{\Delta}{\Phi,\alpha}}{\frac{\Psi}{\Gamma}\big|\frac{\Delta}{\Phi}}\ cut_f$$

Using the structural rules above, the conditions 1 - 8 for the determinants for conjunction are turned into rules. As an illustration, this will be done for conditions 1 - 4, since an equivalent thing can be done for conditions 5 - 8 using the rules for the other diagonal of B-entailment. In this, "condition 1" will be shortened to "1" and so forth.

Condition 1 can be turned into a rule using the rule cut_t twice:

$$\dfrac{\dfrac{\frac{\Psi}{\Gamma}\big|\frac{\Delta,\alpha}{\Phi} \quad \dfrac{}{\frac{\Psi}{\alpha,\beta,\Gamma}\big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}\ 1}{\frac{\Psi}{\beta,\Gamma}\big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}\ cut_t \quad \frac{\Psi}{\Gamma}\big|\frac{\Delta,\beta}{\Phi}}{\frac{\Psi}{\Gamma}\big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}\ cut_t \qquad \Rightarrow \qquad \dfrac{\frac{\Psi}{\Gamma}\big|\frac{\Delta,\alpha}{\Phi} \quad \frac{\Psi}{\Gamma}\big|\frac{\Delta,\beta}{\Phi}}{\frac{\Psi}{\Gamma}\big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}\ Rule\ 1$$

Condition 2 can be turned into a rule using the rules cut_t and $weak$:

$$\dfrac{\dfrac{\frac{\Psi}{\Gamma}\big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}{\frac{\Psi}{\alpha,\Gamma}\big|\frac{\Delta,\beta,\alpha\wedge\beta}{\Phi}}\ weak \quad \dfrac{}{\frac{\Psi}{\alpha,\alpha\wedge\beta,\Gamma}\big|\frac{\Delta,\beta}{\Phi}}\ 2}{\frac{\Psi}{\alpha,\Gamma}\big|\frac{\Delta,\beta}{\Phi}}\ cut_t \qquad \Rightarrow \qquad \dfrac{\frac{\Psi}{\Gamma}\big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}{\frac{\Psi}{\alpha,\Gamma}\big|\frac{\Delta,\beta}{\Phi}}\ Rule\ 2$$

Condition 3 can be turned into a rule using the rules cut_t and $weak$:

$$\dfrac{\dfrac{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}{\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta,\alpha,\alpha\wedge\beta}{\Phi}}\,weak \qquad \dfrac{}{\frac{\Psi}{\beta,\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}}\,3}{\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}}\,cut_t \qquad\Rightarrow\qquad \dfrac{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}{\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}}\,\mathit{Rule}\ 3$$

Condition 4 can be turned into two different rules using cut_t and $weak$:

$$\dfrac{\dfrac{\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha,\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}\,weak \qquad \dfrac{}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta,\alpha,\beta}{\Phi}}\,4}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}\,cut_t \qquad\Rightarrow\qquad \dfrac{\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}\,\mathit{Rule}\ 4.1$$

$$\dfrac{\dfrac{\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha,\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}\,weak \qquad \dfrac{}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta,\alpha,\beta}{\Phi}}\,4}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}\,cut_t \qquad\Rightarrow\qquad \dfrac{\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}\,\mathit{Rule}\ 4.2$$

These five rules can be reduced to fewer rules, and it will be shown how Rule 2 and Rule 3 can be derived from the Rule 4.1 and Rule 4.2.

Theorem 3. *Rule 2 and Rule 3 can be derived from Rule 4.1 and Rule 4.2.*

Proof. To derive Rule 2, take $\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha\wedge\beta}{\Phi}$ as a hypothesis, then a proof of $\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}$ maybe be produced as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta,\beta}{\Phi}}\,ref_t}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta,\beta}{\Phi}}\,\mathit{Rule}\ 4.2 \qquad \frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\beta}{\Phi}}\,cut_t}{\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}}\,weak$$

To derive Rule 3, take $\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha\wedge\beta}{\Phi}$ as a hypothesis, then a proof of $\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta,\beta}{\Phi}$ maybe be produced as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}}\,ref_t}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}}\,\mathit{Rule}\ 4.1 \qquad \frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha\wedge\beta}{\Phi}}{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}}\,cut_t}{\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta,\beta}{\Phi}}\,weak$$

□

To show that these rules are sound and complete for FDE, it will be shown how these three rules (Rule 1, Rule 4.1 and Rule 4.2) are inter-derivable with the $\Rightarrow_{\mathbf{t}}$ rules for $\wedge$ of the sequent calculus for FDE presented by Blasio [Blasio, 2016a] which has been proved to be sound and complete. A similar thing can be done for the $\Rightarrow_{\mathbf{f}}$ rules, and dually for disjunction.

Logical Rules:

Conjunction

$$\frac{\frac{\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi,\alpha} \qquad \frac{\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi,\beta}}{\frac{\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi,\alpha\wedge\beta}} \Rightarrow_{\mathbf{f}} \wedge \qquad \frac{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha}{\Phi} \qquad \frac{\Psi}{\Gamma}\Big|\frac{\Delta,\beta}{\Phi}}{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha\wedge\beta}{\Phi}} \Rightarrow_{\mathbf{t}} \wedge \qquad \frac{\frac{\alpha,\beta,\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\alpha\wedge\beta,\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi}} \wedge \Rightarrow_{\mathbf{f}} \qquad \frac{\frac{\Psi}{\alpha,\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}} \wedge \Rightarrow_{\mathbf{t}}$$

Disjunction

$$\frac{\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta}{\Phi} \qquad \frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha\vee\beta,\Gamma}\Big|\frac{\Delta}{\Phi}} \vee \Rightarrow_{\mathbf{t}} \qquad \frac{\frac{\alpha,\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi} \qquad \frac{\beta,\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\alpha\vee\beta,\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi}} \vee \Rightarrow_{\mathbf{f}} \qquad \frac{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha,\beta}{\Phi}}{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha\vee\beta}{\Phi}} \Rightarrow_{\mathbf{t}} \vee \qquad \frac{\frac{\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi,\alpha,\beta}}{\frac{\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi,\alpha\vee\beta}} \Rightarrow_{\mathbf{f}} \vee$$

Negation

$$\frac{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\alpha}{\Phi}}{\frac{\neg\alpha,\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi}} \neg \Rightarrow_{\mathbf{f}} \qquad \frac{\frac{\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi,\alpha}}{\frac{\Psi}{\neg\alpha,\Gamma}\Big|\frac{\Delta}{\Phi}} \neg \Rightarrow_{\mathbf{t}} \qquad \frac{\frac{\alpha,\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\Gamma}\Big|\frac{\Delta,\neg\alpha}{\Phi}} \Rightarrow_{\mathbf{t}} \neg \qquad \frac{\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\Gamma}\Big|\frac{\Delta}{\Phi,\neg\alpha}} \Rightarrow_{\mathbf{f}} \neg$$

In way of illustration, these rules $\Rightarrow_{\mathbf{t}} \wedge$ and $\wedge \Rightarrow_{\mathbf{t}}$ will be shown to be inter-derivable with Rule 1, Rule 4.1 and Rule 4.2. Rule $\Rightarrow_{\mathbf{t}} \wedge$ and Rule 1 are already identical.

Theorem 4. *Rule $\Rightarrow_{\mathbf{t}} \wedge$ is inter-derivable from Rule 4.1 and Rule 4.2.*

Proof. To derive $\Rightarrow_{\mathbf{t}} \wedge$, take $\frac{\Psi}{\alpha,\beta,\Gamma}\Big|\frac{\Delta}{\Phi}$ as a hypothesis, then a proof of $\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}$ maybe be produced as follows.

$$\frac{\frac{\Psi}{\alpha,\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha\wedge\beta,\beta,\Gamma}\Big|\frac{\Delta}{\Phi}} \textit{ Rule } 4.1$$

To derive Rule 4.1 and Rule 4.2, take $\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta}{\Phi}$ and $\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta}{\Phi}$ as hypothesis, then a proof of $\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}$ maybe be produced as follows.

$$\frac{\frac{\frac{\Psi}{\alpha,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha,\beta,\Gamma}\Big|\frac{\Delta}{\Phi}} \textit{ weak}}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}} \wedge \Rightarrow_{\mathbf{t}} \qquad\qquad \frac{\frac{\frac{\Psi}{\beta,\Gamma}\Big|\frac{\Delta}{\Phi}}{\frac{\Psi}{\alpha,\beta,\Gamma}\Big|\frac{\Delta}{\Phi}} \textit{ weak}}{\frac{\Psi}{\alpha\wedge\beta,\Gamma}\Big|\frac{\Delta}{\Phi}} \wedge \Rightarrow_{\mathbf{t}}$$

□

Based on this sequent calculus, additional structural rules can be added to simulate logics by reducing the truth values from 4 to fewer, which are reflexivity rules.

$$\frac{}{\frac{\alpha}{\cdot}\,\Big|\,\frac{\alpha}{\cdot}}\ ref_p \qquad\qquad \frac{}{\frac{\cdot}{\alpha}\,\Big|\,\frac{\cdot}{\alpha}}\ ref_q$$

The rule (ref_p) restricts the values of 4 eliminating $\bot$, and the rule (ref_q) similarly eliminates $\top$ instead. Notice that by eliminating both these values it is possible to recover a two-valued logic, since by eliminating $\top$ and $\bot$ from **4**, only **t** and **f** remain. By using the same logical rules, and varying the structural rules, it is possible to simulate different logics which are based on the truth values of **4** of a subset of these value. More on this will be presented in Intra-theoretic pluralism (section 3.3 bellow).

In this chapter the framework of B-entailment has been presented, along with its cognitive attitude semantics. B-entailment was presented as being a framework which can simulate known consequence relations in the literature and it was argued that this is a broad framework that can accommodate many known logics. A method for creating a sequent calculus for logics based on refinements of B-entailment was presented and the topic of non-determinism was also explored, contributing to show how general B-entailment is. In the next chapter, the topic of logical pluralism is revisited and B-entailment is analyzed in terms of the varieties of pluralism presented in Chapter One, namely GTT pluralism, intra-theoretical pluralism, proof-theoretical pluralism and eclectic pluralism.

3 Pluralism Revisited

This chapter discusses how the framework of B-entailment squares with the varieties of pluralism presented in Remarks on Pluralism (Chaper One above), namely GTT pluralism, proof-theoretical pluralism, intra-theoretical pluralism and eclectic pluralism. GTT pluralism, even limited to Tarskian consequence relations, can still accommodate logical consequence relations simulated by B-entailment, and some cases are proposed which does just this. B-entailment also fits well with proof-theoretical pluralism (both Restall's and Paoli's) because it can accommodate many logical consequence relations by use of said structural rules, and the issue of meaning variance does not arise. With regard to intra-theoretical pluralism, B-entailment seems to not fit, because it makes use of structural rules for its sequent calculus, which Hjortland rejects. This problem, however, is explored here with the conclusion that it might not be such a disadvantage. Beyond this, within an eclectic pluralism, the framework presented in the previous chapter offers a way to explore the issue of vagueness within a formal language and regarding what a consequence relation amounts to.

3.1 GTT pluralism

Beall and Restall's pluralism is one based on accepting more than one way of settling the notion of consequence relation, also called a precisification, and following the GTT[1] they can accept different precisifications. For Beall and Restall, validity is taken to be preservation of truth, and logical consequence for them must be reflexive, transitive and monotonic.

> The given kinds of non-transitive or irreflexive systems of 'logical consequence' are logics by courtesy and by family resemblance, where the courtesy is granted via analogy with logics properly so called. Non-transitive or non-reflexive systems of 'entailment' may well model interesting phenomena, but they are not accounts of logical consequence. One must draw the line somewhere and, pending further argument, we (defeasibly) draw it where we have. We require

[1] This is the slogan thesis that "an argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion".

> transitivity and reflexivity in logical consequence. We are pluralists. It does not follow that absolutely anything goes. [Beall and Restall, 2006, p. 91]

Beall and Restall's position on non-Tarskian notions of entailment is to reject them up-front. There is, however, more to the B-entailment framework than simply being transitive or reflexive (or not), since it turns a consequence relation into a multi-dimensional structure and allows for more forms of reflexivity and transitivity, as can be seen by the rules ref_t, ref_f, ref_p and ref_q and also the cut rules for transitivity. The consequence relations simulated by B-entailment can then be "multi-dimensionally" Tarskian with the appropriate choice of generalization of Tarskian properties.

Beall and Restall also require that the consequence relations which are part of their variety of pluralism be in some sense necessary, formal and normative. Again, by necessary the authors just means that "logical consequence applies under any conditions whatsoever" [Beall and Restall, 2006, p. 16], by formal they mean that it is topic-neutral, and by normative they mean that it is clear how one might reason wrongly and commit a mistake. The consequence relations presented here are all of these, by the general guidelines presented by them.

It is then possible to present cases using B-entailment with a notion of validity associated with the case, for which arguments are said to be valid. For example, let case$_t$ and case$_f$ be cases in which arguments are valid according to ref_t and cut_t, and ref_f and cut_f, respectively. As they require, the truth conditions for the logical particles are the same ones, which are given by their rules in a sequent calculus. The intuition behind case$_t$ is that of preservation of truth from premisses to conclusion, and case$_f$ of falsity preservation from conclusions to premisses.

By generalizing the Tarskian framework, B-entailment opens up GTT pluralism to other options of consequence relations which adhere to their requirements of necessity, formality, normativity, reflexivity and transitivity. The cases which become possible in this way also go beyond the examples explored of classical, intuitionist and relevant logic, and more can be said about cases with more truth-values. The cases possible with B-entailment do not yet have a model theoretical semantics as do the ones explored by GTT pluralism, but this is not a requirement of this kind of pluralism, as the question of what counts as a case was left open. It seems plausible that a pluralism with a cognitive attitude semantics is also possible.

3.2 Proof-theoretical pluralism

Restall's pluralism [Restall, 2014] is one which features one language, one proof system, but more than one consequence relation. To accomplish this, a sequent calculus is presented in which the rules for the connectives are the same, and structural rules are used to establish a consequence relation. Different structural rules give rise to different consequence relations which co-inhabit the logic. As was indicated in Chapter Three above, there is a sequent calculus that can accommodate more than one consequence relation with the same rules for the connectives, with the addition of structural rules giving rise to multiple consequence relations in the framework of B-entailment.

Restall's pluralism as it was presented can encompass classical logic, intuitionistic logic and dual-intuitionistic logic, motivating GTT pluralism from a formal perspective. In the framework presented here, these three logics can be expressed as forms of B-entailment, given the appropriate structural rules just as it was done by Restall, plus additional restrains on the form of B-entailment. For all three, the following restrains are used:

Definition 21. (One-dimensional consequence) $\Gamma \models \varphi$ iff $\frac{\cdot}{\Gamma}\Big|\frac{\varphi}{\cdot}$ and $\frac{\Gamma}{\cdot}\Big|\frac{\cdot}{\varphi}$

For the intuitionist and dual-intuitionist, rules might also be added to restrict multiple conclusion framework to a single conclusion framework, in which it's possible to have a intuitionist calculus, and also restrict multiple premisses framework to a single premisse framework, to accommodate a dual-intuitionist calculus.

The scheme presented in the previous chapter is able to accommodate Restall's and following similar guidelines be a generalization of it. "[T]he schemes determining the meaning of connectives are unitary" [Restall, 2014, p. 12], that is, after determining the meaning of the connectives through the appropriate sequent rules, plurality comes by the application of structural rules. With this in mind, it is possible to use B-entailment to express logics such as K3, LP and FDE, which in the framework of B-entailment can be determined by the same logical rules by using different structural rules as a way to establish these different consequence relations. Take the logical and structural rules from the last chapter for FDE, and add the following structural rules, which are able to simulate K3 and LP:

$$\frac{}{\frac{\cdot}{\varphi}\Big|\frac{\cdot}{\varphi}}\ ref_q \quad \text{for K3} \qquad\qquad \frac{}{\frac{\varphi}{\cdot}\Big|\frac{\varphi}{\cdot}}\ ref_p \quad \text{for LP}$$

The rule ref_q is meant to restrict the values from **4** that a sentence φ can take by eliminating $\perp$, and the rule ref_p does a similar restriction on the value $\top$. This then restricts the value from **4** in two different ways, so that the middle value from LP and K3 can be taken to be $\top$ and $\perp$, respectively. Restall's pluralism is an attempt to motivate GTT pluralism's semantic approach to pluralism from a proof-theoretical perspective, which is the same that happens with B-sequents and cognitive attitude semantics.

Regarding Paoli's pluralism, B-entailment offers a framework in which one needs not to worry with the translations between logics, since there is a single formal language for more than one logic. Logics express with B-entailment also have the same similarity type, and different structural rules give rise to different notions of consequence. In the framework of B-entailment there can be rival logics with different provable sequents which give rise to pluralism. While B-entailment itself is a neutral framework, it can be used to see how more than one logic can have the same vocabulary and yet differ in regards to the notion of consequence.

The approach shown herein contributes to the proof-theoretical pluralism, because it increases the number os logics that can be seen as part of this approach. Shapiro's main concern with Restall's pluralism is that it does not encompass all logics of interest. The present thesis contributes to the discussion by adding logics in the mix, albeit not all. By way of the procedure presented in last chapter, it's also possible to simulate many more logics than those presented here. The possibility of using B-entailment as a framework to work with other logics is left open, but as a pluralism within proof-theory, B-entailment is interesting on its own.

3.3 Intra-theoretic pluralism

Hjortland's pluralism is one in which there can be two consequence relations on the same language. In this variety of pluralism, there are two distinct consequence relations with the same interpretation for the connectives that treat the truth values differently with respect to validity. The example given by Hjortland is that of K3 and LP. Hjortland says this framework only works for a limited number of cases. Furthermore, it is interesting to see if B-entailment can fit with this pluralism and also provide more examples.

Hjortland's anti-exceptionalism for logic predicts that "one theory be a cumulative extension of another, but that structure is preserved" and "the preceding theory survives as a special, limiting case of the new theory" [Hjortland, 2014, p. 19]. The framework of B-entailment is an extension of the framework he has presented, would it be accepted in this kind of pluralism?

Take the matrices for K3 and LP, namely, $\mathfrak{M}^{K3} = \langle\{1, i, 0\}, \{1\}, \mathcal{O}\rangle$ and $\mathfrak{M}^{LP} = \langle\{1, i, 0\}, \{1, i\}, \mathcal{O}\rangle$ (where $\mathcal{O}$ is the set of truth functions of the connectives of $\mathcal{S}$, which is a recursively formed language formed by atomic formulas and any set of propositional connectives from Chapter 1 in Cognitive attitude semantics). The consequence relation for these logics is such that:

$\Gamma \vDash_{K3} A$ iff there is no v such that $v(\Gamma) \subseteq \{1\}$ and $v(A) \in \{i, 0\}$.

$\Gamma \vDash_{LP} A$ iff there is no v such that $v(\Gamma) \subseteq \{1, i\}$ and $v(A) = \{0\}$.

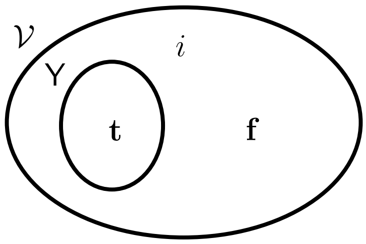

Figure 5: Matrix for K3.

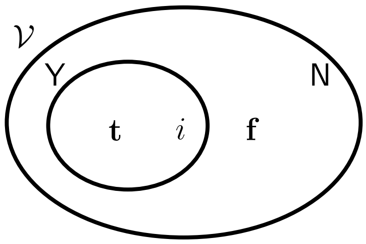

Figure 6: Matrix for LP.

B-entailment can simulate these consequence relations by adding to the calculi for FDE: for K3, add (ref_q) and for LP add (ref_p) instead. For K3, $\top$ is eliminated from **4** and so $i = \{\bot\}$ and for LP, $\bot$ is eliminated and $i = \{\top\}$. For both of them, the form of the argument is as in t-entailment, so the accepted values are preserved.

Note that eliminating these truth-values, the truth tables for K3 and LP become more determined than the truth-table for FDE, in the following way:

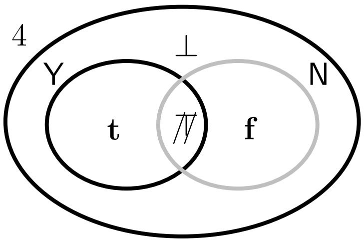

Figure 7: Symmetrical Matrix for K3 with ref_q.

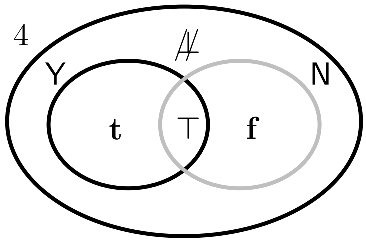

Figure 8: Symmetrical Matrix for LP with ref_p.

$\wedge$	$\mathbf{f}$	$\perp$	$/\!\!\top\!\!/$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$/\!\not\mathbf{f}/$	$\mathbf{f}$
$\perp$	$\mathbf{f}$	$\perp$	$/\!\not\mathbf{f}/$	$\perp$
$/\!\!\top\!\!/$	$/\!\not\mathbf{f}/$	$/\!\not\mathbf{f}/$	$/\!\!\top\!\!/$	$/\!\!\top\!\!/$
$\mathbf{t}$	$\mathbf{f}$	$\perp$	$/\!\!\top\!\!/$	$\mathbf{t}$

$\vee$	$\mathbf{f}$	$\perp$	$/\!\!\top\!\!/$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\perp$	$/\!\!\top\!\!/$	$\mathbf{t}$
$\perp$	$\perp$	$\perp$	$/\!\not\mathbf{t}/$	$\mathbf{t}$
$/\!\!\top\!\!/$	$/\!\!\top\!\!/$	$/\!\not\mathbf{t}/$	$/\!\!\top\!\!/$	$/\!\not\mathbf{t}/$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$/\!\not\mathbf{t}/$	$\mathbf{t}$

	$\neg$
$\mathbf{f}$	$\mathbf{t}$
$\perp$	$\perp$
$/\!\!\top\!\!/$	$/\!\!\top\!\!/$
$\mathbf{t}$	$\mathbf{f}$

Truth tables for K3.

$\wedge$	$\mathbf{f}$	$/\!\!\perp\!\!/$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$/\!\not\mathbf{f}/$	$\mathbf{f}$	$\mathbf{f}$
$/\!\!\perp\!\!/$	$/\!\not\mathbf{f}/$	$/\!\!\perp\!\!/$	$/\!\not\mathbf{f}/$	$/\!\!\perp\!\!/$
$\top$	$\mathbf{f}$	$/\!\not\mathbf{f}/$	$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$	$/\!\!\perp\!\!/$	$\top$	$\mathbf{t}$

$\vee$	$\mathbf{f}$	$/\!\!\perp\!\!/$	$\top$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$/\!\!\perp\!\!/$	$\top$	$\mathbf{t}$
$/\!\!\perp\!\!/$	$/\!\!\perp\!\!/$	$/\!\!\perp\!\!/$	$/\!\not\mathbf{t}/$	$/\!\not\mathbf{t}/$
$\top$	$\top$	$/\!\not\mathbf{t}/$	$\top$	$\mathbf{t}$
$\mathbf{t}$	t	$/\!\not\mathbf{t}/$	$\mathbf{t}$	$\mathbf{t}$

	$\neg$
$\mathbf{f}$	$\mathbf{t}$
$/\!\!\perp\!\!/$	$/\!\!\perp\!\!/$
$\top$	$\top$
$\mathbf{t}$	$\mathbf{f}$

Truth tables for LP.

As an example of how using appropriate refinements on B-entailment to simulate K3 and LP with the sequent rules from Chapter 2 in Sequent calculus give rise to different logics, it is shown how the law of excluded middle (LEM) fails in K3 and is valid in LP, as well as explosion[2] holds in K3 and fails in LP.

Theorem 5. *LEM is valid in LP.*

Proof.

$$\dfrac{\dfrac{\dfrac{}{\frac{\alpha}{\cdot} \,\Big|\, \frac{\alpha}{\cdot}}\; ref_p}{\frac{\cdot}{\cdot} \,\Big|\, \frac{\alpha,\neg\alpha}{\cdot}}\; \Rightarrow_t \neg}{\frac{\cdot}{\cdot} \,\Big|\, \frac{\alpha\vee\neg\alpha}{\cdot}}\; \Rightarrow_{\mathbf{t}} \vee$$

□

[2] The principle of explosion states that once a contradiction is reached, anything follows.

Theorem 6. *Explosion is valid in K3.*

Proof.

$$\dfrac{\dfrac{}{\begin{array}{c|c} \dfrac{\cdot}{\alpha} & \dfrac{\cdot}{\alpha} \end{array}}\; ref_q}{\begin{array}{c|c} \dfrac{\cdot}{\alpha,\neg\alpha} & \dfrac{\cdot}{\cdot} \end{array}}\; \neg\Rightarrow_{\mathbf{t}}$$

□

To see that LEM is not valid in K3, suppose $\text{⅄}s{:}\alpha$ and $\text{И}s{:}\alpha$. Then $\text{И}s{:}\neg\alpha$ and $\text{⅄}s{:}\neg\alpha$ (by the clauses 3 and 4 from Cognitive attitude semantics), thus $\text{⅄}s{:}\alpha \vee \neg\alpha$ (from clause 11), and the argument is invalid. To see that explosion is not valid in LP, suppose $\mathsf{Y}s{:}\alpha$ and $\mathsf{N}s{:}\alpha$. Then $\mathsf{N}s{:}\neg\alpha$ and $\mathsf{Y}s{:}\neg\alpha$ (by the clauses 1 and 2 from Cognitive attitude semantics), thus $\mathsf{Y}s{:}\alpha \wedge \neg\alpha$ (from clause 5), and the argument is invalid.

B-entailment can thus be used to simulate K3 and LP. This does not mean, however, that this framework fits into intra-theoretical pluralism. In order to achieve this simulation structural rules were needed to restrict the values $\top$ and $\bot$, which is something undesirable from Hjortland's point of view. Pluralism should be achieved without use of structural rules [Hjortland, 2014].

What Hjortland claims to be doing with his 3-sided sequent calculus is varying the consequence relation and also maintaining the same structural rules. In his approach, LP and K3 are defined based on the form of derivable sequents. He also proposes an *n*-sided sequent calculus beyond the 3-sided one he presented, and points to a 4-sided calculus that can encompass also FDE. With *B*-entailment, consequence is defined for four values and by changing the structural rules, LP and K3 can be defined by the addiction of strutural rules which eliminte one of the four values (either $\top$ or $\bot$). The difference between these two approaches seems to be that one makes explicit the role structural restrictions play on the definition of logical consequence.

Buying into Hjortland's requirement that the different consequence relations cannot come from use of different structural properties, it is still possible to look at the framework of *B*-entailment with interest. Through its many dimensions, expressed by the definitions of *t*-consequence, *f*-consequence, *p*-consequence and *q*-consequence, it is still possible to simulate different consequence relations on the same language in a similar way as to what Hjortland does for his 3-sided calculus. This move to reach a pluralism seems to be just a different way to look at a structure, so the difference is still structural, which is another reason why Hjortland's claim that structural differences play no role in his variety of pluralism is strange. This issue, however, remains to be studied further in future works. Apart from this disagreement on the use of structural rules, B-entailment is a framework which offers to intra-theoretical pluralism the possibility of expressing the notions of consequence of t-consequence, f-consequence, p-consequence and q-consequence, which are also interesting on their own.

3.4 Ecclectic pluralism

Shapiro's logical pluralism [Shapiro, 2014] encourages one to keep an open mind when it comes to logic and also defends that the notion of logical consequence is vague. This section explores the framework of B-entailment in light of this take on pluralism.

Shapiro starts embracing "a form of folk-relativism concerning logic, insisting that logical consequence and validity are relative to a mathematical theory or structure (or a type of structure)" [Shapiro, 2014, p. 96]. From this point of view, B-entailment is a structure which logical consequence can be relative to. What is proposed here is to take logical consequence to be relative to an aspect of B-entailment, meaning choosing from one of the forms of entailments it can simulate.

Shapiro's view on the vagueness of a consequence relation can be viewed as as a defense against Hjortland's rejection of structure as a way to differentiate consequence relations. Shapiro holds that a consequence relation is relative to something, and accepting at least two such ways in which something can be relative would be enough to make one a pluralist. If logical consequence is seen as relative to a particular choice of structural rules (which reflect on a choice to make the matrix of a logic more restrictive, for example), there it is no problem with this way of achieving a pluralism.

The pluralism proposed with the B-entailment framework takes a lesson from Restall and Paoli's pluralism in giving minimal conditions for the meaning of the connectives and uses structural rules to refine different consequence relations, while adopting Hjortland's treatment of many-valued logics. Due to its use of structural rules, it does fall short of Hjortland's standard of pluralism. B-entailment has, however, the same flavor as these other two varieties of pluralism, and as such fits into Shapiro's account. In this way, it also suffers from the problems of the varieties of pluralism of this kind, in that it is not as general as a framework as one might hope it to be. It is a step into the right direction, insofar as it further explores the possibilities of a pluralism of this kind.

Another aspect of B-entailment, and perhaps the main one, is its use of a four-place (or two-dimensional) consequence relation. In *Varieties of Logics*, the consequence relations explored are one-dimensional and Tarskian, thus abiding by the transitivity, reflexivity and monotonicity properties, were expressed in the single conclusion framework, as well as being only concerned about the preservation of truth. These are all ways to limit the notion of logical consequence and to show that, even with these restrains, there is still room for logical pluralism. In this thesis, the aim was to open the field and explore a pluralism among consequence relations which are less restrictive. As Beall and Restall put it, perhaps these more unusual consequence relations should not be called such, in as much as they stray too far from what one might wish to include as a consequence relation.

At the beginning of this discussion, a logical consequence relation was taken to be a relation between two sets of formulas, such that the conclusion follows from the premisses in case "when all premises belong to some designated set of truth values, the conclusion belongs to some possibly distinct designated set" [Chemla et al., , p. 1]. This definition allows more than just the preservation of truth, but also the preservation of falseness. It seems that treating the preservation of some set of values deemed true instead of values deemed false is arbitrary. Choosing to work with a cognitive attitude semantics seems to overlook this dispute between "truth" and "falseness" and treat all kinds of preservation equally. In Chapter 2, moreover, other forms of entailment were explored, which dealt with the backward preservation of truth values, as well as with two distinguished sets of values, and thus opened up different possibilities for the notion of consequence.

Accepting the tolerant approach proposed by Shapiro, it's possible to use B-entailment to explore consequence relations that have different forms of transitivity and reflexivity, as well as non-Tarskian consequence relations. In defense of these kinds of entailment, Shapiro states:

> There is no one, monolithic thing that goes by the name of 'validity' or 'logical consequence.' Those terms are polysemous or, perhaps better, they express cluster concepts. There are a number of distinct notions that go by those names, none of which can lay claim to being *the* notion of logical consequence.

[Shapiro, 2014, p. 205]

B-entailment seems interesting because its structure can express distinct notions of entailment.

Another important point to Shapiro's take on logical pluralism is vagueness. He has argued that in some situations the logical particles have the same meaning, and also that in some situations there is meaning-shift. This approach to vagueness can be explored formally by looking at the non-deterministic aspect of the B-entailment framework. As was seen, depending on how precise one wishes to make the meaning of a connective, the truth-function can be more deterministic or less. Once this kind of semantics is allowed, it's possible to have a formal counterpart to the vagueness which is found on some terms.

Shapiro seems to leave an invitation for logicians to "play in the spirit of tolerance" [Shapiro, 2014, p. 204]. In light of this, the framework of B-entailment was presented as a playground in which different entailment relations can be found. B-entailment offers a lot for a logical pluralist, especially once structure is accepted as an interesting aspect and explored in its two dimensions.

In this chapter, it was shown how the framework of B-entailment fits into the pluralism of Beall and Restall, by presenting a case$_B$ for cognitive attitude semantics, and Restall and Paoli, by arguing that through a sequent calculus for this framework it's possible to use structural rules to define different consequence relations on the same logical rules. This framework does not, however, quite fit the bill according to the pluralism of Hjortland. The question was left open whether the use of structural rules goes indeed against intra-theoretical pluralism, since both approaches seem very similar to each other. Finally, by accepting Shapiro's eclectic approach to logic, it was argued that B-entailment can offer interesting insights into the nature of logical consequence and vagueness.

Conclusion

The thesis has aimed at discussing logical pluralism and the framework of B-entailment. Logical pluralism is the view that more than one logic is good or correct. This can mean many things. In this thesis, focus was given to pluralisms that aim at avoiding the meaning variance problem, which arises when the specification of logics rely on different languages. The varieties of pluralism explored were those of Shapiro, Beall and Restall, Restall alone, Paoli and Hjortland. In Chapter One, a discussion of what is logical pluralism was presented, along with the characterization of the varieties of pluralism that are featured here.

In Chapter Two, B-entailment was presented formally. The semantics of cognitive attitudes was introduced, and B-entailment was presented through generalized semantical matrices and different notions of consequence were shown to be expressed using B-entailment. By exploring the multi-dimensional consequence relation of B-entailment, other notions of entailment were shown to be a particular case of B-entailment. A brief exposition of non-determinism was also presented, which helped pave the way to a brief discussion on vagueness. Finally, a method which allows logics that can be expressed on the framework of B-entailment to be characterized by a sequent calculus was presented, with FDE being used as the example. It is noteworthy that B-entailment itself is a neutral framework, which does not advocate for any specific logic, but can instead be trimmed one way or another to simulate other consequence relations which might be desired.

Chapter Three returned to the topic of logical pluralism, presenting a discussion of how B-entailment can be of interest to the different varieties of pluralism presented, namely those of Shapiro, Beall and Restall, Restall alone, Paoli and Hjortland. With regards to the pluralisms of Beall and Restall, it was found that B-entailment can accommodate their demand of reflexivity and transitivity, and even further, that it offers a way to see them in different forms, depending on how one looks at the structure of the consequence relation in question.

Regarding Restall's and Paoli's varieties of pluralism, the framework presented here works in a similar way and can then be seen as an advocate for the kind of logical pluralism that elaborates on a plurality of logical consequence relations. Through the method presented of generating sequent rules, it's possible to use B-entailment to simulate logics in a similar way as was done by Restall and Paoli. With regards to Restall's pluralism, the sequent calculus for B-entailment complements its semantic counterpart. As to Paoli's pluralism, it seems enough that B-entailment uses structural rules as a way of differentiating consequence relations, and this is a fit.

Intra-theoretical pluralism seems to be the most challenging one, because it rejects the move of using structure to justify changing the consequence relation. This rejection does not, however, seem to be justified. Still under this imposition, the framework of B-entailment yet offers something, as the different dimensions present different notions of entailment, namely, t-entailment, f-entailment, p-entailment and q-entailment.

The framework of B-entailment was also argued to be of interest to Shapiro's pluralism, since it elaborated on the vagueness of what is a consequence relation and also the vagueness of the expression "same meaning as". Once Shapiro's eclectic approach to logic is accepted, entailment relations such as the ones mentioned here become an option, and with B-entailment they can be seen as equally natural as the usual consequence relation of preservation of truth. The non-deterministic semantics of B-entailment also offers some in the way of dealing with the vague meaning of connectives, as it allows for truth functions that are indeterminate in their output.

Future work

The framework offered by B-entailment is a rich field that still offers much to be explored. This section will introduce some such topics which are of interest to the discussion on logical pluralism as a suggestion of future work.

The canonical logical consequence relation can be seen as a one-dimensional structure, while B-entailment can be seen as two-dimensional. While the canonical consequence deals with two truth-values, B-entailment deals with four such values. In the same way as B-entailment is a more general framework than the canonical consequence (which is one particular case of B-entailment), it's possible to think of another framework that is more general than B-entailment in the same way. This structure would be a three-dimensional one that deals with eight truth-values. This generalization could be performed again and again. Such move for B-entailment was accompanied by a motivation to distinguish rejection from non-acceptance. If a similar motivation exists for a further generalization, this path is there to be explored. This generalization could offer material for the discussion of logical pluralism in the same way as B-entailment did, by offering a new view of what is logical consequence.

In Chapter 2, some specific consequence relations were explored using B-entailment, but that was not an exhaustive survey of the topic. This framework still offers other consequence relations to be studied, which could be fed into the discussion of pluralism. This further study could add weight to the expressive power of B-entailment and promote the B-entailment framework as a more viable framework for logical consequence than the uni-dimensional notion of consequence.

It is also interesting to explore further intuitionist logic in this new framework, specially regarding the new vocabulary of acceptance and rejection. It is important to investigate that the philosophical motivations of intuitionist logic are not lost in this formal framework. Also in this topic, it is of particular interest to the pluralist to investigate Prawitz's ecumenical system.

Another project which arises from this discussion of logical pluralism is to explore further the issue of structure and Hjortland's rejection of the use of structural rules to change the notion of logical consequence. He argues that this leads to the presence of meaning variance concerning the notion of consequence, and for him this is a bad thing. It Chapter Three it was hinted that perhaps the change of structural rules with B-entailment is not so different from what would be done in the 4-sided calculus that Hjortland projects. It remains to be seen how structure plays a role in changing notions of consequence and if this indeed leads to meaning variance.

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